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THE ELEMENTS OF PLANE GEOMETRY.

BY

RICHARD P. WRIGHT,

TEACHER OF MATHEMATICS IN UNIVERSITY COLLEGE SCHOOL, LONDON.

WITH A PREFACE

BY

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RICHARD P. WRIGHT.

University College School, Gower Street, W.C.

ON FORM

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LESSONS ON FORM

FOR THE USE OF TEACHERS AND PUPILS
IN ELEMENTARY SCHOOLS

BY

RICHARD P. WRIGHT

TEACHER OF MATHEMATICS IN UNIVERSITY COLLEGE SCHOOL, LONDON
AUTHOR OF 'THE ELEMENTS OF PLANE GEOMETRY'

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PREFACE.

THE course pursued in this Elementary Treatise is founded on Pestalozzi's system of teaching Geometry. This system which has long been practised in Holland, Switzerland, and Germany, is no doubt often unconsciously adopted by teachers of elementary classes in English schools, although there are at present no recognised text-books suggestive of the method. Its chief characteristic is to move step by step from the simplest notions to the more advanced geometrical truths.

The lessons are designed for schools where time does not admit of the pupils reading the advanced text-books on Pure Geometry in ordinary use. Notwithstanding their elementary character, they will be found to cover a good deal of ground in that subject. It is moreover hoped, and is in fact intended, that they should be suggestive to teachers of those amplifications which beginners always need in an abstract subject like Geometry.

In a few cases it was not convenient to adhere to a strictly logical sequence of the articles. The author is aware that, considering the nature of the subject and the mental discipline which results from a sound course of it, this is a serious defect, yet in the present instance it is more than counterbalanced by certain advantages.

UNIVERSITY COLLEGE SCHOOL:
December 1881.

CONTENTS.

PART I.

INTRODUCTION.

PAGE
Solids, surfaces, and lines 1

CHAPTER I.

Definitions—Angles—Quadrilaterals—Solids 3

CHAPTER II.

The Straight Line—Straight Lines joining a given number of Points—Relative Positions of Lines—Points of Intersection 11

CHAPTER III.

Angles—Parallel Lines and Secants 19

CHAPTER IV.

Triangles—Equality and Similarity of Triangles . . . 34

CHAPTER V.

Quadrilaterals: Square, Rhombus, Rectangle, Parallelogram, and Trapezoid 41

CHAPTER VI.

Areas: Unit of Measure of Length, Unit of Area, Areas of Parallelograms 44

PART II.

CHAPTER VII.

PAGE

Theorem of Pythagoras—Metrical Properties of Triangles—Projection of a Line—Regular Polygons—Angles of Regular Polygons 49

CHAPTER VIII.

Rhombus—Pentagon—Hexagon, &c. 60

CHAPTER IX.

Interior Polygons 65

CHAPTER X.

Star-Polygons—Intersection of Figures—Equivalent Figures—The Circle—The Circle and Straight Line—Chords—Arcs—Sectors—Segments—Angles in Circles—Tangents—External and Internal Contact of two Circles—Inscribed and Circumscribed Polygons—Circle inscribed in and described about a Triangle—Metrical Properties of inscribed and described regular Polygons in terms of Diameter of Circle—Theorem of Archimedes—Ratio of Circumference to Diameter—Area of Circle 68

LESSONS ON FORM.

PART I.

INTRODUCTION.

1. In the first lesson on this subject, the attention of the pupils should be drawn to the different *forms* of any familiar objects in the class-room. The seats and desks may be used for this purpose with advantage. When such objects are compared with a set of mathematical solids, which the teacher ought to have at hand, the difference between *regular* and *irregular* bodies will be at once seen. This comparison leads to a knowledge of the outward forms of objects.

2. Every solid has *three* dimensions, namely, *length*, *breadth*, and *thickness*. Instead of thickness the term *depth* or *height* is sometimes used. For instance, we speak of the thickness of a beam of timber, the depth of a well, and the height of a pyramid.

3. Where a solid is separated from the surrounding space, we have what is called the *boundary* of the solid.

The boundary of a solid is termed its *surface*. When the surface of a solid consists of several distinct *parts* or *planes*, each such part is called a *face* or *plane*.

Surfaces are either plane or curved.[1] A surface has *two* dimensions only, length and breadth.

4. The boundary of a surface is called a *line.* The boundary of a plane surface is termed a *straight line;* that of a curved surface, a *curved line.* Lines can have *one* dimension only, length. Lines terminate in *points.* Points have no dimensions whatever; but merely position. And it must always be remembered that the visible points which we draw on the black-board merely denote the positions of the mathematical points just defined.

---

[1] There are surfaces partly plane and partly curved, termed *mixed surfaces,* but teachers will not be called upon to consider such at present.

CHAPTER I.

DEFINITIONS.

5. In the *triangular pyramid,* or *four-faced* solid, we have a solid with the least number of plane faces possible. This (producing it), which is a *cube,* has six faces. The faces of the triangular pyramid are bounded by three lines, those of the cube by four.

6. Two straight lines AB, AC (fig. 1), drawn from the same point A, form an *angle.* The straight lines AB, AC are the *sides* of the angle and the point A is called its *vertex.* We may obtain a very clear notion of the magnitude of an angle by supposing one of its sides AC (fig. 2) to at first coincide with the other side AB, and then to turn about the point A as one of the legs of a pair of compasses may be turned about its joint. As the side AC moves in the direction of the arrow, the point A remaining fixed, we see that the angle BAC (which is at first a very small or *acute angle*) gradually increases in magnitude. The straight line AC during its rotation around the point A acquires a position AD such that it is said to be

Fig. 1.

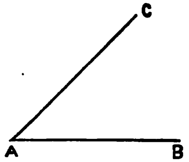
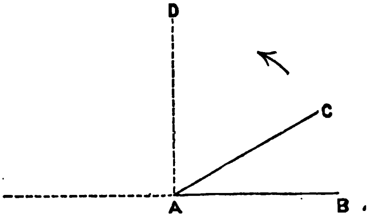


Fig 2.

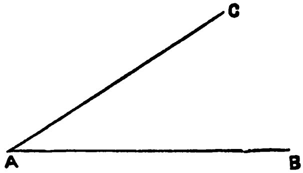


perpendicular to A B. The angle B A C, extended to B A D, is then called a *right-angle.* By continuing this rotation the angle B A C becomes greater than a right angle, and is then what is termed an *obtuse angle.* The straight line A C will, during its rotation around the point A, have one position such that it will be in the same straight line with A B, it will then have turned through two right angles.

Remembering the comparison of an angle to the space comprised between the legs of a pair of compasses, since an angle becomes larger by opening or closing its sides, it is evident that the magnitude of an angle is altogether independent of the *length* of the sides.

7. If the points C and B (fig. 3) of the sides of the angle be joined, we form a figure in which there are three angles;

FIG. 3.



such a figure is termed a *triangle.* The faces of the first solid (triangular pyramid, or tetrahedron) to which we referred (5) are triangles.

8. In the six-faced solid or cube we have already noticed that each face is bounded by four lines.

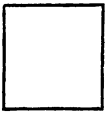
Figures bounded by four lines are called *quadrilaterals.* With four straight lines the following quadrilaterals can be formed: 1°, A *square*; 2°, A *rhombus*; 3°, A *rectangle* or *oblong*; 4°, A *parallelogram*; 5°, A *trapezoid*; 6°, A *trapezium.*

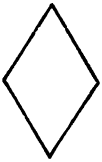
A square is a quadrilateral whose sides are all equal, and whose angles are right angles.

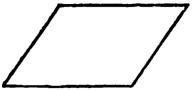
A rhombus is a quadrilateral whose sides are all equal and whose angles are *not* necessarily right angles.

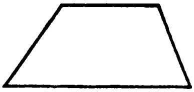
A rectangle is a quadrilateral whose opposite sides are equal and whose angles are right angles.

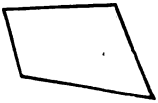
Every quadrilateral whose opposite sides are *parallel* (that is to say, would never meet however far they may be produced) is a parallelogram. A trapezoid is a quadrilateral which has only two opposite sides parallel. It is *right-angled* when one of its non-parallel sides is perpendicular to the other two. It is *isosceles* when its two non-parallel sides are equal. All other quadrilateral figures are termed trapeziums.











9. The four-faced solid or triangular pyramid is bounded by four equal and equilateral triangles. Its faces are bounded by 4 × 3 lines, or six *edges*; 4 × 3 plane angles and four solid angles. *Half the number of lines which bound the faces of the solid is equal to the total number of its edges.* For each edge is the intersection of two faces; in other words, it requires two coincident lines to form an edge. The number of plane angles of the faces of any solid is always equal to the total number of lines which bound those faces. A solid angle must be formed by at least three faces which meet in the same point. When three or more faces meet at the same point, the same number of plane angles necessarily meet at that point; hence the number of solid angles or *corners* in any solid can always be obtained by dividing the total number

of plane angles by the number of faces which form a corner of the solid. Thus in the triangular pyramid (tetrahedron) there are twelve plane angles, and three faces form a corner; hence it has four solid angles.

10. The faces of a cube, or *hexahedron*, are squares; each face is bounded by four lines: hence twenty-four lines are required to form the different faces of a cube. Now we have already stated that two coincident lines form an edge (9); therefore there are twelve edges in a cube, four of which are vertical and eight horizontal. There are $6 \times 4 = 24$ plane angles; and since the meeting of three plane angles at the same point forms a corner of the cube, we have $24 \div 3 = 8$ solid angles or corners.

11. The eight-faced solid or *octahedron* is bounded by eight equal and equilateral triangles. It follows from what has been already stated (9) that its faces are formed by $8 \times 3 = 24$ lines. It contains $\frac{24}{2} = 12$ edges, and $8 \times 3 = 24$ plane angles. If we examine one of its corners, we see that *four* and not three faces meet in the same point. Therefore its number of solid angles is $\frac{24}{4} = 6$.

12. The twelve-faced solid, or *dodecahedron*, is bounded by twelve equal faces, called *pentagons*, which are equilateral and equiangular. Now, since a pentagon has five sides, it follows that $12 \times 5 = 60$ lines are required to form the faces of the dodecahedron. It consequently has $\frac{60}{2} = 30$ edges, $12 \times 5 = 60$ plane angles, and $\frac{60}{3} = 20$ solid angles.

13. The twenty-faced solid, or *icosahedron*, is bounded by twenty equal and equilateral triangles. Thus its twenty faces are bounded by $20 \times 3 = 60$ lines. It contains $60 \div 2 = 30$ edges, and has $\frac{60}{5} = 12$ solid angles: for five faces meet at the same point, and form a solid angle.

The five regular geometrical solids just described, *viz.*, the *tetrahedron*, *hexahedron*, *octahedron*, *dodecahedron*, and *icosahedron*, are termed the *Platonic bodies*, because they were first described by Plato (400 B.C.). Besides these five, there can be no other solids bounded by like equal and regular plane figures, and whose solid angles are all equal.

14. The next set of solids of which we shall speak are *prisms*. A prism is a solid bounded by faces, two of which are always equal, similar and parallel, and all the rest are parallelograms. We shall call that face on which the prism stands its *base*, which will always be one of the two equal, similar, and parallel faces of the prism—that is, one of its two ends. The face opposite to the base will be termed the *upper face* of the prism. In many works on Solid Geometry these two faces are frequently termed the *bases* of the solid. Prisms are distinguished by the figures of their bases or ends. When the ends are triangles they are called *triangular prisms*; when the ends are square, *square prisms*; when the ends are pentagonal, *pentagonal prisms*, etc.

If the side of a prism is perpendicular to its base, it is a *right prism*; if not, an *oblique prism*. In a right prism, each lateral, or side edge, is equal to the height of the prism. The side or lateral faces of a right prism are rectangles. In the triangular prism there are $(2\times3)+(3\times4)=6+12=18$ lines, which bound the base and upper face and the side faces of the solid. It has, therefore, $18\div2=9$ edges, and $18\div3=6$ solid angles or corners.

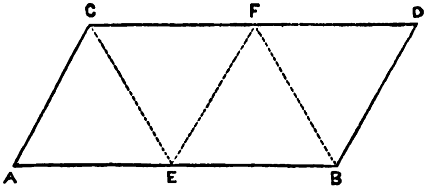
15. In the *quadrangular prism*—that is, a prism whose base is a square—there are $(2\times4)+(4\times4)=8+16=24$ lines or twelve edges, and consequently twenty-four plane angles, and therefore $\frac{24}{3}=8$ corners. Such a prism is also termed a *parallelopiped*.

The number of edges, plane and solid angles of an hexagonal and an octagonal prism respectively, may be readily found by the pupil.

16. It may be remarked that models of all the regular solids can easily be made of cardboard. The pupil is advised to commence with the tetrahedron, which is bounded by four equal equilateral triangles. Two parallel lines should be drawn in pencil on cardboard, and between them four equilateral triangles described side by side, as in fig. 4.

Cut out the parallelogram A B D C, but only cut C E, F E, F B half through the cardboard. If the cardboard be folded

FIG. 4.



at D F, F E, and E B, and A D, C B be brought together so as to form an edge of the solid, a model of a tetrahedron will be obtained.

17. *Pyramids.* The side or lateral faces of a pyramid meet at the same point, which is termed the vertex of the pyramid. The base may be any regular polygon. The side faces of a regular pyramid are all equal triangles.

Pyramids are distinguished by the figures of their bases, thus we have a *three-faced* or *triangular pyramid.* It is more frequently called a tetrahedron. A *four-faced* or *quadrangular pyramid.* Such is the pyramid of Cheops. A *five-faced* or *pentagonal pyramid.* A *six-faced* or *hexagonal pyramid.* An *eight-faced* or *octagonal pyramid.*

18. The three-faced pyramid requires $3 \times 3 + 3 = 9 + 3 = 12$ lines to form its four distinct faces. It has $12 \div 2 = 6$ edges, 12 plane angles, and consequently $12 \div 3 = 4$ solid angles.

19. The four-faced pyramid requires $4 \times 3 + 4 = 12 + 4 = 16$ lines to form its four triangular faces and square base. It has $16 \div 2 = 8$ edges, and 16 plane angles. Now the solid angle, which is the vertex of this pyramid, is formed by the four side faces meeting at the same point. But each solid angle at the base is formed by *three* faces meeting at the same point. Hence from the 16 plane angles we must deduct 4, the number which meet at the vertex, and divide

the remainder 12 by 3. We thus obtain 5 as the total number of solid angles or corners in the quadrangular pyramid.

20. In order to find the number of solid angles in any pyramid, we deduct from its total number of plane angles the number of plane angles which meet at the vertex, and divide the remainder by three. The quotient increased by one (for the solid angle at the vertex) gives the number of solid angles in the pyramid

As an exercise find the number of edges, plane and solid angles respectively in an hexagonal and an octagonal pyramid.

Ex.—If the base of a regular pyramid is a *quindecagon* —that is, has 15 sides—how many solid angles are there in the pyramid?

21. The *cylinder* is a solid whose two ends are equal and parallel circles. Besides these, which are frequently termed its two bases, it has but one curved face. We may say it is formed by four curved lines, and has therefore two edges. It has no solid angles, because we have already seen that it requires at least *three* sides or faces to form a solid angle or corner. Cylinders are either *right* or *oblique*. A *right* cylinder is such that the straight line passing through the centres of its two ends or bases is perpendicular to the latter. When this is not the case, the cylinder is termed *oblique*. The straight line which is supposed to pass through the centre of the ends of the cylinder is called its *axis*. In a *right* cylinder the axis is the perpendicular height of the cylinder. But in an oblique cylinder the perpendicular height of it is the perpendicular distance between its two parallel bases.

A right cylinder can be formed, or generated, by the revolution of a rectangle around one of its sides which remains fixed.

22. A *cone* is a solid having a circle for its base, and tapering off to a point, which is called its vertex. When the *axis* of the cone, that is, the straight line through its centre,

is perpendicular to its base, it is a *right cone*; when such is not the case, it is *oblique*. A *right* cone can be formed, or *generated*, by the revolution of a right-angled triangle around its perpendicular which remains fixed. Besides its base the cone has *one* curved face, and also *one* curved edge. The vertex cannot be termed a solid angle: for the latter can only be formed by three or more planes meeting in one point.

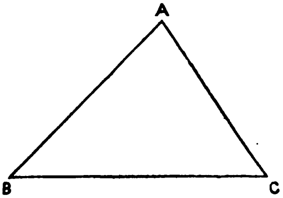
23. The *sphere* or *ball* is a solid bounded by one face, every point of which is equidistant from a single point within the surface, called the centre of the sphere. A sphere can be formed or generated by turning a semi-circumference around its diameter.

CHAPTER II.

THE STRAIGHT LINE.

24. BETWEEN two points an infinite number of lines may be drawn; the shortest, however, will be a *straight line.* Two points can be joined by *one* straight line only. If a third point be taken, but not in the same straight line with the other two, and these points be joined by straight lines, a *triangle* is obtained.

FIG. 5.

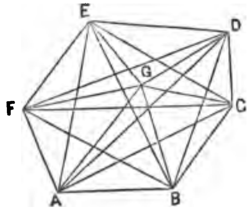


Thus, by drawing straight lines from the point A to B and C, and joining the points B and C by a third straight line, the points A, B, and C are all joined together.

In taking three, four, or more points, care must be taken that no three points are in the same straight line, otherwise we cannot distinguish the right number of straight lines which can be drawn between the points. In order to be certain that each point is joined to every other point, the following regular mode of procedure may be adopted. Suppose, by way of example, that seven points be taken. Let them be denoted by A, B, C, D, E, F, G respectively.

From A (fig. 6), which we will call the first point, draw straight lines to each of the other six points; thus six lines are required to join A to all the other points. From B draw straight lines to the points C, D, E, F and G. For B is already joined to A by the straight line from A to B; therefore from B only five additional straight lines are required to be drawn. Now let us take the point C, which is already joined to A and B, and join it to D, E, F, G; only four additional lines are required. Proceeding in like manner with D, E, F, G, we see that the number of additional lines drawn from each succeeding point, taken in order, is one less than the preceding. When we come to G, which is the seventh point, no additional line is needed; for it is already joined to every other point. It will be readily seen that the number of straight lines proceeding from each point must be the same.

FIG. 6.



25. We have already observed that as we proceed from point to point in regular order, *one line less* is drawn each time. Hence a rule for finding the *greatest number of straight lines which can be drawn between a given number of points*. In the example given, 7 points were taken; 6 lines were drawn from the point A, 5 from B, 4 from C, 3 from D, 2 from E, 1 from F, *none* from G. Thus the total number of lines which can be drawn between 7 points is 21. For $6+5+4+3+2+1=21$. This is an arithmetical series; its *first term* is 6, *last term* 1, *common difference* 1, *number of terms* 6, that is one less than the number of points: the sum

of the series is 21. If 9, 10, 12, or *n* points be taken, for the first case we have $8+7+6+5+4+3+2+1=36$ lines, and similarly for 10, 12, or *n* points. The sum of such an arithmetical series may easily be found by adding the *first* and *last* terms together and multiplying their sum by *half the number of terms.* In our first example, the number of points taken is 7, hence the first term of the series is 6, the last is 1, their sum multiplied by 3, half the number of terms, $=21$.

How many straight lines are required to join 100 points?

26. Length and position of lines with respect to each other.

FIG. 7.

If the straight line A B be placed on C D, so that when the point A is on C the point B coincides with D, then the straight lines A B and C D are said to be *equal.*

FIG. 8.

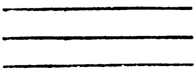


FIG. 9.

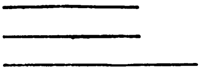
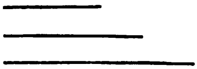


FIG. 10.



If three straight lines be compared, we see that they may be all equal (fig. 8); two equal only (fig. 9); all unequal (fig. 10).

FIG. 11.

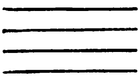


FIG. 12.

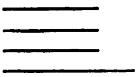
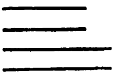


FIG. 13.



If four lines be compared with regard to their length, we see that they may be—

1°. All equal (fig. 11).

2°. Three of the four equal (fig. 12).

3°. Two and two equal (fig. 13).

4°. Two equal and two unequal (fig. 14).
5°. Four unequal (fig. 15).

FIG. 14.

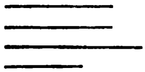
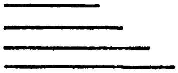


FIG. 15.



Of five given straight lines
1°. All equal (fig. 16).
2°. Four of them equal and one unequal (fig. 17).
3°. Three equal and two equal (fig. 18).

FIG. 16.

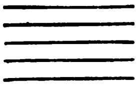


FIG. 17.

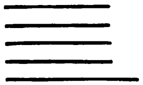


FIG. 18.

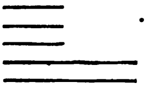


FIG. 19.

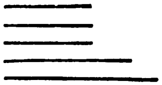


FIG. 20.

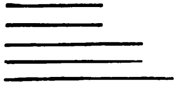


FIG. 21.

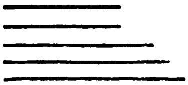
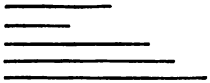


FIG. 22.



4°. Three equal and two unequal (fig. 19).
5°. Of four, two and two equal and one unequal (fig. 20).
6°. Two equal and three unequal (fig. 21).
7°. All unequal (fig. 22).

In a similar manner six and more straight lines may be compared. It is convenient to first suppose all the lines equal, then to separate them by taking one, two, three, &c. away; and with the lines thus separated to observe the various cases which may occur, being careful to avoid repetition.

27. The relative positions of two straight lines in the same plane must be—

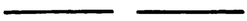
1°. In the same direction (fig. 23).

2°. Equidistant or parallel (fig. 24).

3°. Inclined to one another (fig. 25).

4°. At right angles or perpendicular to one another (fig. 26).

Fig. 23.



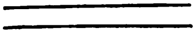


Fig. 24.

Fig. 25.

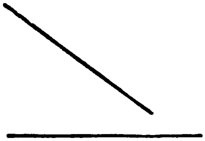
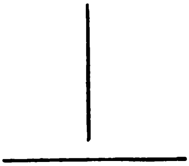


Fig. 26.



Lines are in the same direction when one is on the *production* of the other.

The statement that two straight lines can have four different positions with respect to each other must be understood in a restricted sense. For we know that two lines may be inclined to each other in an infinite number of ways. We say they are inclined when they are neither in the same direction, parallel, nor perpendicular to one another. But we do not indicate their exact inclination to each other.

28. We shall now proceed to consider the relative positions of three and more straight lines.

Fig. 27.

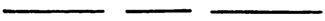
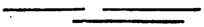


Fig. 28.



1°. Three lines in the same direction (fig. 27).

2°. Two lines in the same direction, and the third parallel to them (fig. 28).

3°. Two lines in the same direction and one perpendicular to them (fig. 29).

4°. Two in the same direction, and one inclined or oblique to them (fig. 30).

FIG. 29.

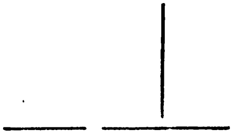
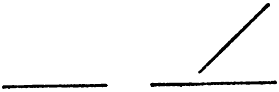


FIG. 30.



5°. The three lines parallel (fig. 31).

6°. Two lines parallel, and the third perpendicular to them (fig. 32).

7°. Two lines parallel, and one inclined or oblique to them (fig. 33).

8°. Two lines perpendicular to one another, and the third inclined to them (fig. 34).

9°. The three lines inclined to one another (fig. 35).

FIG. 31.

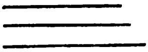


FIG. 32.

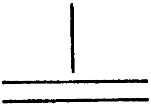


FIG. 33.

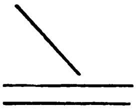


FIG. 34.

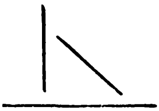
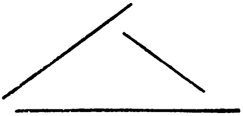


FIG. 35.



Similarly, the relative positions of four, five, six, &c. lines may be found. (These cases are to be given as exercises to the pupils, whose attention should be drawn to the various combinations and the order in which they are taken in the foregoing example, which will serve as a guide in other cases.)

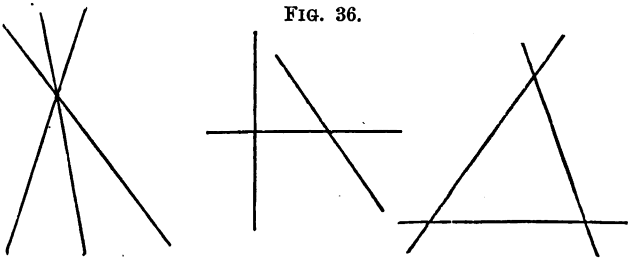
Points of Intersection.

29. Two straight lines inclined to one another will meet if *produced.* If produced beyond the point where they meet, the lines are said to *intersect* or *cut* one another. The point where they cut one another is called their *point of intersection.* Two straight lines intersect or cut one another in one point only. On both sides of the point of intersection the lines are *divergent*—that is, when produced they recede from each other—or, in other words, the distance between them becomes greater and greater.

30. Two straight lines can only intersect in *one* point.

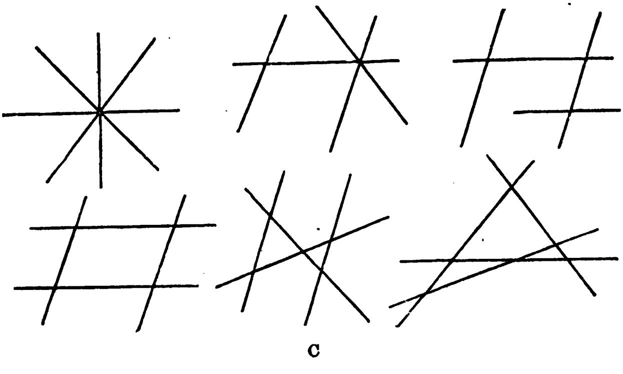
Three straight lines can intersect in *one, two,* or *three* points.

FIG. 36.



31. Four straight lines can intersect in *one, two, three, four, five,* or *six* points.

FIG. 37.



32. It is quite evident that the number of points of intersection of a given number of straight lines must be limited—that is, cannot exceed a certain number. We shall now endeavour to show what that limit is in a few cases which we have chosen as examples.

We already know that two straight lines can only intersect in *one* point. If we take a third straight line it may cut the other two so as to give *no* new point of intersection, *one* or *two*, according as it is drawn through the point of intersection of the other two straight lines, cuts *one* of them only, or *both*.

If we take a fourth straight line and suppose it to cut the first three, it cannot form more than *three* new points of intersection. A fifth line cannot cut the last four in more than *four* new points.

Thus two straight lines can intersect in 1 point only.

Three straight lines in	$1+2=3$	points.
Four	$1+2+3=6$	,,
Five	$1+2+3+4=10$	,,
Six	$1+2+3+4+5=15$	,,

Generally n lines give $1+2+3+4+$ &c. . . . . $n-1$ points. The greatest number of intersections of n straight lines is given by the expression $\frac{n\,(n-1)}{2}$. Or, for the help of those who do not understand the interpretation of this algebraical expression, we may say—Multiply the given number of lines by itself *less* one, and divide the product by 2.

Find the greatest number of intersections which can be formed by 50 straight lines.

Various exercises may be given on this chapter. For instance, the pupils may be required to find the number of points of intersection of a given number of parallel and non-parallel straight lines.

CHAPTER III.

ANGLES. (i.)

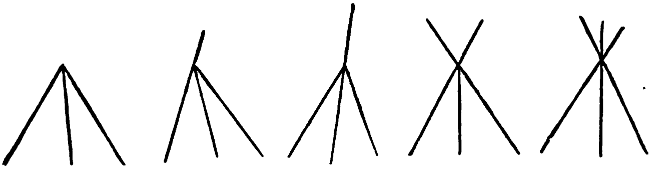
33. If two straight lines, which are not parallel, be produced, they will meet, in which case they form an *angle.*

Two straight lines form *one, two,* or *four* angles.

Three straight lines may be drawn so as to form several angles. We shall now proceed to show by means of figures the various cases which can occur.[1]

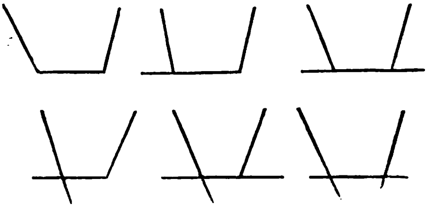
1°. Three straight lines meeting in *one* point form 2, 3, 4, 5, or 6 angles (fig. 38).

FIG. 38.



2°. Three straight lines coming together in *two* points will form 2, 3, 4, 5, 6, or 8 angles (fig. 39).

FIG. 39.



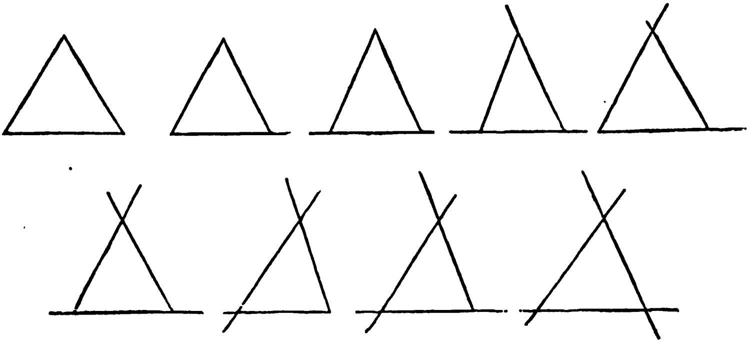
We see that 7 angles cannot be formed when the lines meet in the manner indicated.

---

[1] Angles which overlap one another are not reckoned as separate angles.

Three straight lines coming together in 3 points form 3, 4, 5, 6, 7, 8, 9, 10, and 12 angles (fig. 40).

FIG. 40.



It is impossible to form 11 angles with three straight lines.

The greatest number of angles is formed when the lines are drawn so as to give the greatest number of intersections.

34. We know that 10 straight lines would give $\frac{n(n-1)}{2}$ that is, $\frac{10\times 9}{2}=45$ points of intersection. At each such point there are four angles. Hence the *maximum* or greatest number of angles which can be formed by 10 straight lines is 180.

Ex.—Find the greatest number of angles which can be formed by 20 straight lines.

Angles. (ii.)

35. We have already seen that there are three kinds of angles—*right*, *acute*, and *obtuse*—and that these different angles can be formed by two lines meeting in one point. When two lines meet so that only two angles are formed, it is evident that either both are right angles (fig. 41), or, one obtuse and the other acute (fig. 42). Hence we may con-

clude that when two lines intersect, the four angles thus formed are either all right angles (fig. 43), or two obtuse and two acute (fig. 44).

FIG. 41.

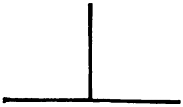


FIG. 42.

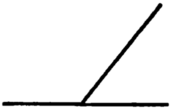


FIG. 43.

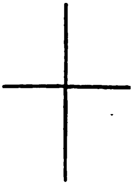
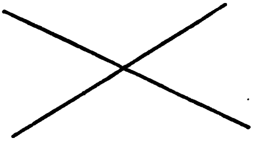


FIG. 44.



36. Three straight lines meeting in one point and forming the least number of angles, *i.e. two*, give the following cases:

1°. Both angles acute (fig. 45).

2°. One angle acute and the other obtuse (fig. 46).

3°. One acute and the other a right angle (fig. 47).

FIG. 45.

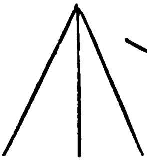


FIG. 46.

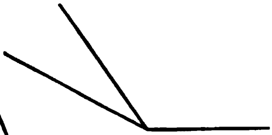
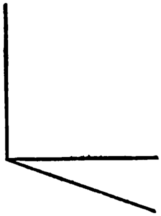


FIG. 47.



Suppose two of the three lines meet so as to form one straight line, the case is then reduced to (35).

When three lines meet in one point and form three angles we have:

1°. The three angles acute (fig. 48).

2°. Two angles acute and one obtuse (fig. 49).
3°. Two obtuse and one acute (fig. 50).
4°. Three obtuse angles (fig. 51).
5°. One right angle and two acute (fig. 52).
6°. One right angle and two obtuse (fig. 53).

FIG. 48.

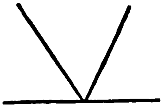


FIG. 49.

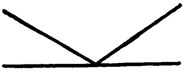


FIG. 50.

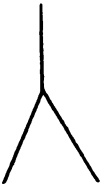


FIG. 51.

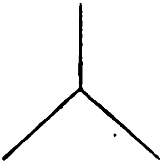


FIG. 52.

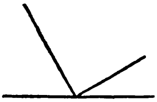
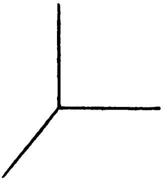


FIG. 53.



If three straight lines form four angles at the same point we have but two cases:

1°. Two right angles, one obtuse and one acute (fig. 54).
2°. Two acute angles and two obtuse angles (fig. 55).

FIG. 54.

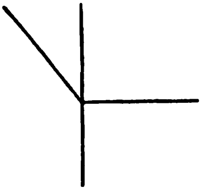
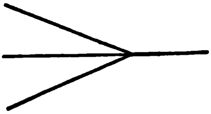


FIG. 55.



If three straight lines form five angles at the same point we have:

1°. One obtuse and four acute angles.

2°. Two obtuse and three acute angles.
3°. One obtuse, one right and three acute angles.
4°. Three right and two acute angles.

If three straight lines form six angles at the same point we have:

1°. Two obtuse and four acute angles.
2°. Two right and four acute angles.
3°. Six acute angles.

Three straight lines coming together in two points and forming two angles present the following cases:

1°. One obtuse and one acute angle.
2°. Two acute angles.
3°. Two obtuse angles.
4°. One right and one obtuse angle.
5°. One right and one acute angle.
6°. Two right angles.

Three straight lines meeting in two points and forming three angles give:

1°. One obtuse and two acute angles.
2°. Two obtuse angles and one acute angle.
3°. One right, one obtuse and one acute.
4°. Two right angles and one acute angle.
5°. Two right angles and one obtuse angle.

Three straight lines meeting in two points and forming four angles give:

1°. Two obtuse and two acute angles.
2°. Two right angles, one obtuse and one acute angle.
3°. Four right angles.

Three straight lines meeting in two points and forming five angles give:

1°. Two obtuse and three acute angles.
2°. Three obtuse and two acute angles.
3°. One right angle, two obtuse and two acute angles.
4°. Four right angles and one obtuse angle.

5°. Four right angles and one acute angle.
6°. Five right angles.

Three straight lines meeting in two points and forming six angles, give :

1°. Three obtuse and three acute angles.
2°. Two right, two obtuse and two acute angles.
3°. Four right, one obtuse and one acute angle.
4°. Six right angles.

Three straight lines meeting in two points and forming eight angles give :

1°. Four obtuse and four acute angles.
2°. Four right, two obtuse and two acute angles.
3°. Eight right angles.

Three straight lines meeting in three points and forming three angles :

1°. Three obtuse angles.
2°. Two obtuse angles and one acute angle.
3°. Two acute angles and one right angle.

Three straight lines meeting in three points and forming four angles :

1°. Two obtuse and two acute angles.
2°. One obtuse and three acute angles.
3°. Two right and two acute angles.
4°. One right, one obtuse and two acute angles.

Three straight lines meeting in three points and forming five angles :

1°. Two obtuse and three acute angles.
2°. Three obtuse and two acute angles.
3°. One right, two obtuse and two acute angles.
4°. Two right, two obtuse and two acute angles.

Three straight lines meeting in three points and forming six angles :

1°. Three obtuse and three acute angles.
2°. Two obtuse and four acute angles.

3°. One right angle, two obtuse and three acute angles.
4°. Two right, two obtuse and two acute angles.
5°. Four right and two acute angles.

The pupils are expected to draw figures corresponding to the foregoing examples.

Parallel Lines and a Secant. (iii.)

37. Three straight lines, two of which are parallel, meeting in two points and forming two angles give:

1°. Two acute angles (fig. 56).
2°. Two obtuse angles (fig. 57).

FIG. 56.

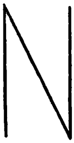


FIG. 57.

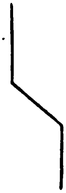


FIG. 58.

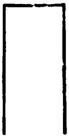
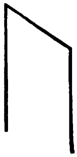


FIG. 59.



3°. Two right angles (fig. 58).
4°. One obtuse and one acute angle (fig. 59).

Three straight lines, two of which are parallel, meeting in two points and forming three angles :

FIG. 60.

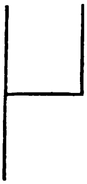


FIG. 61.

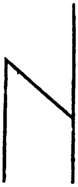
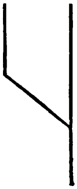


FIG. 62.



1°. Three right angles (fig. 60).

2°. Two acute and one obtuse angle (fig. 61).
3°. Two obtuse and one acute angle (fig. 62).

Three straight lines, two of which are parallel, forming four angles:

1°. Four right angles (fig. 63).
2°. Two obtuse and two acute angles (fig. 64).

FIG. 63.

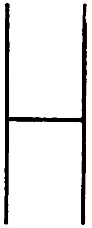
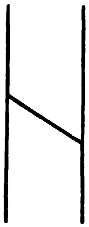


FIG. 64.



Two parallels and a secant forming five angles give:

1°. Five right angles (fig. 65).
2°. Two acute and three obtuse angles (fig. 66).
3°. Three acute and two obtuse angles (fig. 67).

FIG. 65.

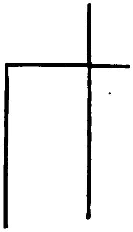


FIG. 66.

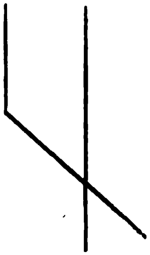
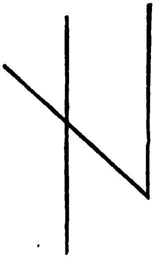


FIG. 67.



Two parallels and a secant forming six angles:

1°. Six right angles (fig. 68).
2°. Three acute and three obtuse angles (fig. 69).

Two parallels and a secant forming eight angles :
1°. Eight right angles (fig. 70).
2°. Four acute and four obtuse angles (fig. 71).

FIG. 68.

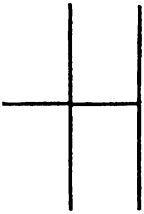


FIG. 69.

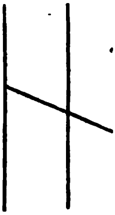


FIG. 70.

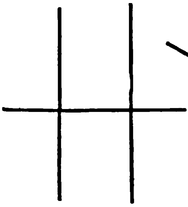
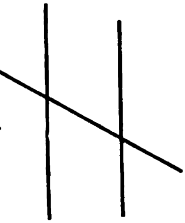


FIG. 71.



Four straight lines, two of which are parallel, meeting in *two* points and forming three angles :

1°. Two right and one acute angle.
2°. One right and two acute angles.
3°. One obtuse and two acute angles.
4°. One right, one obtuse and one acute.
5°. Two obtuse and one acute angle.
6°. Three acute angles.

Four straight lines, two of which are parallel, meeting in two points and forming four angles :

1°. Three right and one acute angle.
2°. Two right and two acute angles.
3°. One obtuse and three acute angles.
4°. One right, one obtuse and two acute angles.

5°. One right and three acute angles.
6°. Two obtuse and two acute angles.

Four straight lines, two of which are parallel, meeting in two points and forming five angles:

1°. Two acute and three right angles.
2°. One right, one acute and three obtuse angles.
3°. One right, one obtuse and three acute angles.
4°. One acute and four obtuse angles.
5°. Two acute and three obtuse angles.
6°. Two obtuse and three acute angles.
7°. One obtuse and four acute angles.

Four straight lines, two of which are parallel, forming six angles :

1°. Five right and one acute angle.
2°. Four right, one obtuse and one acute angle.
3°. Three right, one obtuse and two acute angles.
4°. Three right and three acute angles.
5°. Two right, two acute, and two obtuse.
6°. Two right, one obtuse and three acute angles.
7°. One right, two obtuse and three acute angles.
8°. Three obtuse and three acute angles.

Four straight lines, two of which are parallel, forming seven angles :

1°. Five right and two obtuse angles.
2°. Five right and two acute angles.
3°. Three right and four acute angles.
4°. Two right, four acute and one obtuse angle.
5°. Two right and five acute angles.
6°. One right, four acute, and two obtuse angles.
7°. Three obtuse and four acute angles
8°. Two obtuse and five acute angles.
9°. Seven acute angles.

Four straight lines, two of which are parallel, meeting in two points and forming eight angles :

1°. Six right, one acute and one obtuse angle.

2°. Four right and four acute angles.
3°. Two right, three obtuse and three acute angles.
4°. Two right, one obtuse and five acute angles.
5°. Three obtuse and five acute angles.
6°. One obtuse and seven acute angles.

Four straight lines, two of which are parallel, meeting in two points and forming nine angles:

1°. Seven right angles and two acute angles.
2°. Five right angles, three acute and one obtuse angle.
3°. Three right angles, four acute and two obtuse angles.
4°. Two right angles, four acute and three obtuse angles.
5°. One right angle, five acute and three obtuse angles.
6°. Six acute and three obtuse angles.
7°. Five acute and four obtuse angles.
8°. Eight acute angles and one obtuse angle.

The case when ten angles are formed under the above conditions is left as an exercise.

The pupils may now be required to draw four straight lines, three of which are parallel, so as to meet in three points and then to indicate the different kinds of angles thus formed. Then take five lines, two of which are parallel, meeting in two points, &c. &c.

Angles. (iv.)

38. When a straight line meets another straight line so that the two adjacent angles thus formed are equal to one another, each of these angles is a right angle. When two straight lines form a right angle they are said to be perpendicular to one another. Let us suppose the sides of a right angle to be movable, like the legs of a pair of compasses. If we close them a little—that is, if we bring the sides closer together—they form an *acute* angle. If they be opened wider than a right angle they form an *obtuse* angle. Thus we see that an acute angle is less, and an obtuse angle is greater, than a right angle.

39. The magnitude of an angle depends on the inclination of the sides to one another, not on their length.

40. It has just been stated that an acute angle is less than a right angle. We can add another angle to an acute angle so as to make the two together equal to *one* right angle. Two such angles are said to be *complementary angles*, or one is termed the *complement* of the other.

If two or more angles have the same complement they are equal.

FIG. 72.

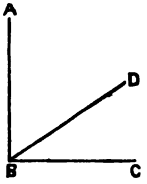
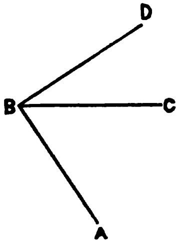


FIG. 73.



At the vertex B of the angle DBC draw BA perpendicular to BC. The angle ABD is the complement of DBC. For it is the *defect* of DBC from a right angle. If instead of drawing the perpendicular to BC it be drawn to BD as in fig. 73, the angle ABC is the complement of DBC, and therefore must be equal to ABD in fig. 72.

41. An obtuse angle is greater than one right angle but less than two. We open the legs of a pair of compasses or the sides of an angle wider than a right angle, and by this means form an obtuse angle. The defect of any angle from two right angles is termed its *supplement*. Thus two angles whose sum is equal to two right angles are called *supplementary angles*. The supplement of any angle is formed by producing either of its two sides beyond the vertex.

42. We shall now prove that the angles formed by one straight line with another and its continuation are together equal to two right angles.

First proof.—We know that two straight lines can only cut one another in one point, and cannot form more than four angles. The two straight lines may intersect at right angles, which will be the case if, in fig. 74, we suppose the line CD to be turned around the point O until it is perpendicular to AB, as in fig. 75. The four angles at the point O, fig. 74, are together equal to the four right angles at the point O′, fig. 75. If we now cut both figures along AB or CD and take away half of each, or two right angles, then since the halves of equals are equal, it is evident that the remaining two angles of fig. 74 are together equal to the remaining two right angles of fig. 75.

FIG. 74.

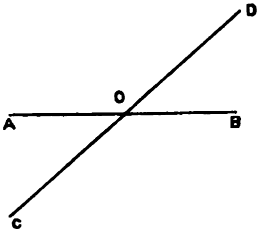
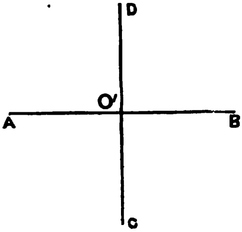
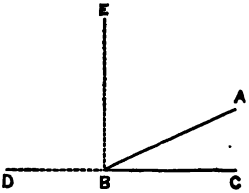


FIG. 75.



43. *Second proof.*—It has been already stated (40) that an angle and its complement are equal to one right angle.

FIG. 76.

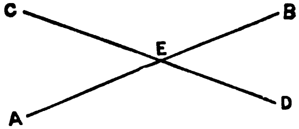


Draw an acute angle ABC, and form its supplement ABD by producing CB beyond the vertex. At the point B erect a perpendicular, BE to BC. ABE is the complement of ABC.

Hence ABC + ABE = one right angle
ABD − ABE = one right angle
ABC + ABD = two right angles

44. When two straight lines (AB, CD, fig. 77) cut one another they form four angles at the point of intersection E,

FIG. 77.



of which the opposite angles are equal. For AEC is equal to BED because they have the same supplement BEC, and the angle AED is equal to the angle BEC because they have the same supplement BED. The equal angles are on opposite sides of the straight line AB or CD, for CE, or part of the straight line CD, must make the same inclination with AB that the remaining part DE does, and therefore the angles CEA and BED are equal.

45. In fig. 78 we have two parallel lines AB, CD, cut by a third line EF, which is called a *secant.* These three straight lines form eight angles, four of which are at each point of intersection. The angles which are between the parallels are called *interior* angles, the other angles are termed *exterior* angles.

46. But besides being called *interior* and *exterior* angles, they are also distinguished in the following manner :

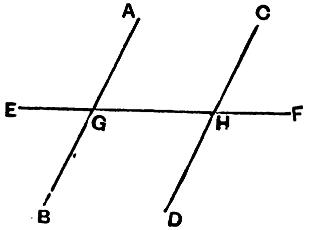
AGH and GHD } *Alternate interior* angles.
BGH and CHG }

AGE and FHD } *Alternate exterior* angles.
EGB and CHF }

EGB and GHD } *Corresponding* angles on the same side of the
FHD and HGB } secant.

EGA and GHC } *Corresponding* angles on the same side of the
AGH and CHF } secant.

FIG. 78.



47. Because AB is parallel to CD, the secant EF is equally inclined to both (44), hence the *corresponding* angles are equal. For the same reason, the *alternate interior* angles are equal, and likewise the *alternate exterior* angles. For the angle EGB is equal to its opposite angle AGH, and EGB and GHD are equal, being corresponding angles; therefore the alternate interior angles AGH and GHD are equal. The angles AGH and CHG are supplementary; for AGH and AGE are supplementary, but AGE and CHG are equal; therefore AGH and CHG are supplementary.

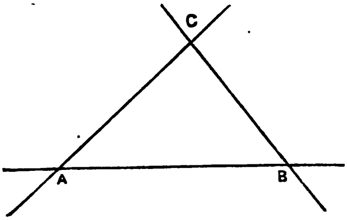
48. Again, AGE and BGH are equal, as opposite angles at the point G of the two intersecting straight lines AB, EF; but BGH is equal to DHF, as corresponding angles, therefore the alternate exterior angles AGE and FHD are equal.

CHAPTER IV.

TRIANGLES.

49. THREE straight lines meeting in such a manner that the extremities of each are joined to an extremity of each of the other two form a figure which is termed a *triangle.* If three straight lines intersect in three distinct points they inclose a space which is called a triangle.

FIG. 79.



The straight lines AB, BC, CA (fig. 79) are the *sides* of the triangle. Each angle formed by two sides is an *angle* of the triangle. AB is called the base, the points A, B, C, are the *vertices.* The angles formed by the base and sides respectively are termed the *angles at the base,* and the third angle is called the *vertical angle.*

50. A triangle cannot have more than *one* right angle, for if we attempt to form a second right angle, two sides, instead of meeting, become parallel, and consequently would never meet.

A triangle in which we have a right angle is termed a *right-angled* triangle (fig. 80).

A triangle which has all its angles acute is called an *acute-angled* triangle (fig. 81).

A triangle which has an obtuse angle is called an *obtuse-angled* triangle (fig. 82).

FIG. 80.

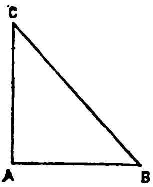


FIG. 81.

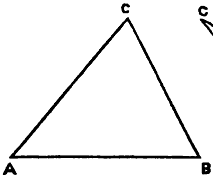
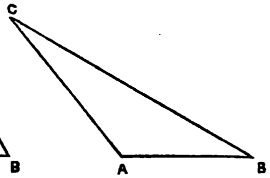


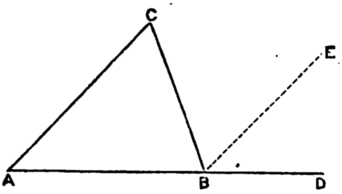
FIG. 82.



51. The three angles of a triangle are always equal to two right angles.

To prove this statement we will produce the side AB to D, and draw BE parallel to AC. We know that the angles

FIG. 83.



at the point B on the same side of AD are together equal to two right angles, therefore the angles ABC+CBE+EBD=2 right angles. The angles EBD and CAB are equal, being corresponding angles. And the alternate-interior angles ACB and CBE are also equal; therefore the three angles CAB, ACB, and ABC are equal to the three angles EBD, EBC, and ABC. But EBD, EBC, and CBA are together equal to two right angles; therefore the three angles of the triangle ABC are equal to two right angles.

52. A right angle being equal to 90°, the three angles of

any triangle, being together equal to two right angles, are equal to 180°.

In a right-angled triangle, if the magnitude of one of the acute angles be known, the other acute angle can be found by subtracting the one known from 90°. For the angles of any triangle are together equal to two right angles; but one of the angles is a right angle, therefore the other two are together equal to a right angle, that is, to 90°. In *acute-* or *obtuse-angled* triangles we must know the magnitude of two of its angles in order to find the third.

53. Three straight lines may be equal, unequal, or *two* only of the three may be equal to one another and the third unequal. Consequently a triangle may have *three*, *two*, or *none* of its sides equal to one another.

FIG. 84.

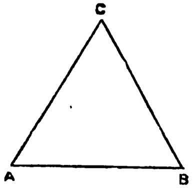


FIG. 85.

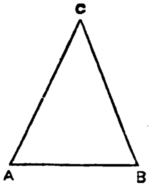
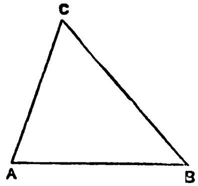


FIG. 86.



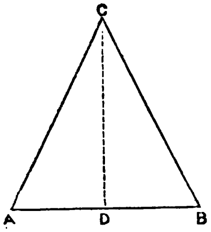
54. Triangles are distinguished according to the relation of their sides as well as of their angles (50). A triangle is *equilateral* when its three sides are equal in length (fig. 84). It is termed an *isosceles* triangle when two of its sides only are equal (fig. 85). When its three sides are unequal it is termed a *scalene* triangle (fig. 86). This latter term, however, is but rarely used.

55. If a perpendicular CD be drawn from the vertex C of the isosceles triangle ABC on the base AB (fig. 87), the triangle will be cut into two equal right-angled triangles, ACD, BCD. Because the angle BCD will be equal to the angle ACD (52). Hence, if the triangle BCD is placed on the triangle ACD, so that the point B coincide with the point A and the straight line BC with AC, then the triangles

will coincide with one another, therefore the side BD will be equal to AD.

We thus see that the perpendicular from the vertex of an isosceles triangle 1° divides the triangle into two equal parts; 2° bisects the base AB; 3° bisects the vertical angle ACB.

FIG. 87.



56. In an isosceles triangle, if the magnitude of one of its angles be known the magnitudes of the others are readily found. For example, let the angle CAB (fig. 87) be 36°, the angle ABC is also 36°. Now we have already shown that the three angles are together equal to two right angles, or 180°; that is

$$\begin{array}{rcl} \mathrm{CAB}+\mathrm{CBA}+\mathrm{ACB} & = & 180^\circ \\ \mathrm{CAB}+\mathrm{CBA} & = & 72^\circ \\ \hline \mathrm{ACB} & = & 108^\circ \\ \hline \end{array}$$

When the vertical angle of an isosceles triangle is a right angle, the three angles are at once known; for each of the equal base angles is half a right angle, *i.e.* 45°.

The magnitude of an angle of an equilateral triangle is one-third of two right angles; that is to say, one-third of $180^\circ = 60^\circ$.

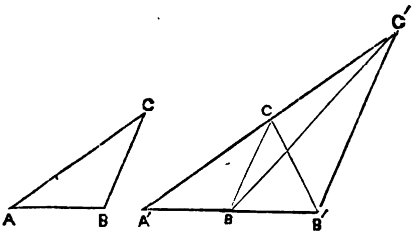
Equality and Similarity of Triangles.

57. In art. 55 we saw that the perpendicular CD bisects the base AB of the isosceles triangle ABC (fig. 87); therefore AD is equal to DB, and AC is equal to CB, because the triangle

is isosceles. By cutting the triangle along CD, and placing one part upon the other so that the point B falls on A and the straight line AD on DB, the two triangles ADC, BDC, will coincide with one another in every respect, because the sides and angles of the one are equal respectively to the sides and angles of the other. Triangles which coincide with one another are said to be *equal*. In equal triangles both the sides and angles are equal. When the three angles of one triangle are equal respectively to the three angles of another, but the sides of the one are not equal to the corresponding sides of the other, the two triangles are said to be *similar*.

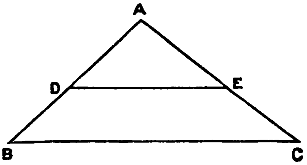
58. Let us consider the case in which the sides of one triangle are a certain number of times (n), greater than the corresponding sides of the other. Take, for example, fig. 88, in which each side of A'B'C' is twice the length of each corresponding side of ABC. By placing the triangle ABC on A'B'C' so that A is on A', and the straight line AB on A'B', also the straight line AC on A'C', the side BC will be parallel to B'C'. The triangles A'BC, B'BC are equal (73), and the triangles A'BC, C'BC are equal (73), therefore the triangles B'BC, C'BC are equal, therefore (73) BC is parallel to B'C'. A similar mode of proof holds so long as A'B' &c. contain AB &c. an integral number of times exactly. We shall *assume* (the proof is beyond the scope of the present work) that the same property holds universally, *i.e.*, for all values of n.

FIG. 88.



59. Any straight line drawn parallel to a side of a triangle cuts off a similar triangle; thus, in fig. 89, the straight line DE is drawn parallel to BC, therefore the triangle ADE is similar to the triangle ABC. For DE and BC are parallel, and the straight line AB meets them, therefore the corresponding angles ADE, ABC are equal, and likewise the corresponding angles AED, ACB. The angle BAC is common to the two triangles ABC, ADE, therefore the three angles of the one are respectively equal to the three angles of the other, and consequently the two triangles are similar.

FIG. 89.



60. The following are important consequences of (58): 1°. The sides of similar triangles are proportional. 2°. A straight line parallel to any side of a triangle cuts the other two sides proportionally. Thus in fig. 89:

$$AB : AD = AC : AE = BC : DE$$

$$\therefore AB - AD : AC - AE = AB : AC = AD : AE$$

$$\therefore BD : EC = AB : AC = AD : AE;$$

and $BD : AB - AD = EC : AC - EC$,

i.e. $BD : AD = EC : AE$ and conversely.

61. All equilateral triangles are similar.

By dividing the sides of an equilateral triangle into a number of equal parts the triangle can be divided into a number of equal equilateral triangles similar to the given triangle. The number of triangles into which any equilateral triangle is thus divided is always equal to the square of the number of equal parts into which each side is divided. Thus in fig.

90 each side is bisected, that is, divided into two equal parts; therefore the triangle is divided into four equal equilateral triangles. In fig. 91 each side is trisected, that is, divided into *three* equal parts; therefore the triangle is divided into nine equal equilateral triangles. In fig. 92 the number of equal equilateral triangles into which the large equilateral triangle is divided is 16, because each of its sides is divided into four equal parts. It may be inferred from this, that one large equilateral triangle can always be constructed from a perfect square number of equal equilateral triangles.

FIG. 90.

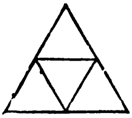


FIG. 91.

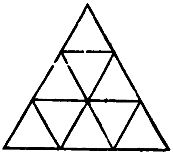
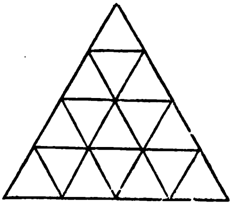


FIG. 92.



CHAPTER V.

QUADRILATERALS.

62. The different figures bounded by four straight lines have been already spoken of. We know how to draw a square, rhombus, rectangle, parallelogram and trapezoid.

Fig. 93.

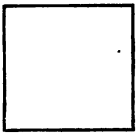
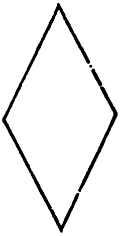


Fig. 94.



In these figures there are parallel sides. In a square there are two pairs of parallel sides (fig. 93), so also in the rhombus (fig. 94), rectangle (fig. 95), and parallelogram (fig. 96). In fact these figures are *parallelograms*, for their opposite sides are parallel.

Fig. 95.

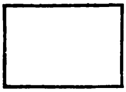
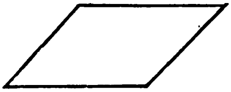


Fig. 96.



63. A trapezoid has only two parallel sides (fig. 97). When one of the two non-parallel sides is at right

angles to the parallel sides, it is termed a *right-angled trapezoid* (fig. 98).

When the two non-parallel sides are equal to one another it is called an *isosceles trapezoid* (fig. 99).

All other quadrilateral figures are called *trapeziums.*

FIG. 97.

FIG. 98.

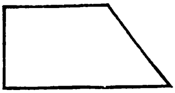
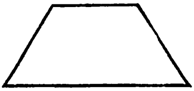
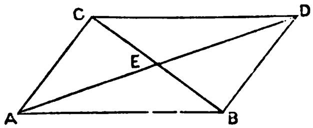


FIG. 99.



64. Every parallelogram can be divided into two *equal triangles* by the straight line which joins two opposite angles. The straight line BC which joins two opposite angles is termed a *diagonal* (fig. 100). We shall proceed to prove that the two triangles into which a diagonal of a parallelogram divides it are equal to one another.

FIG. 100.



65. Because the straight lines AB, CD are parallel, and the secant AD meets them, the alternate interior angles CDA, DAB are equal. And because AC and BD are parallel, and the same secant AD meets them, the alternate interior angles CAD and ADB are equal. We see then that

$$\left.\begin{array}{l}\text{CDA} = \text{DAB}\\ \underline{\text{ADB} = \text{CAD}}\end{array}\right\}\text{; adding these equals together}$$

$$\underline{\text{CDB} = \text{CAB}}$$

Again, because CD and AB are parallel, and the secant CB meets them, the alternate interior angles DCB, CBA are equal. And because AC and BD are parallel, and the same secant CB meets them, the alternate interior angles ACB, CBD are equal. So that we see

$$\left.\begin{array}{l}\text{DCB} = \text{CBA}\\ \underline{\text{ACB} = \text{CBD}}\end{array}\right\}; \text{ adding these equals together}$$

$$\underline{\underline{\text{ACD} = \text{ABD}}}$$

We have now proved that the three angles of one triangle are respectively equal to the three angles of the other, viz. ABC to BCD, ACB to DBC, and CAB to CDB, and we know that the sides AB, AC are equal respectively to CD, DB, and the diagonal CB common; therefore the two triangles, having their sides and angles respectively equal, are equal.

66. In every parallelogram :

1°. A diagonal divides it into two equal triangles.

2°. The opposite sides and angles are equal.

3°. The diagonals bisect one another.

4°. The greater diagonal is opposite, or subtends, the greater angle.

5°. If one angle is a right angle all the other angles are right angles, and the two diagonals are equal.

The last three cases, viz. 3°, 4°, and 5° have not been proved. They may be given as exercises to the pupil.

CHAPTER VI.

AREAS.

67. In order to ascertain the *length* or *linear* distance between two points, the *yard* has been adopted as a *unit* of *lineal measure.* The yard is divided into *three feet,* and the foot is subdivided into twelve *inches.* The multiples of the yard are the *pole* or *perch,* the *furlong* and the *mile.* A pole contains a yard 5½ times, a furlong 220, and a mile 1760 times.

68. The square yard is a *unit of measure* for *areas* For land-measure the multiples of the yard are the *pole,* the *rood* and the *acre.* Very large surfaces, such as whole countries, are expressed in *square miles.*

69. We may remark that lineal measurements are *actually taken*; but square measurements are found by aid of arithmetic. Thus, we can find the area of a *square field* by simply taking the length of one of its sides in yards and squaring the number which denotes that length. We could find its area by actually dividing it into square plots each *one yard square,* and then counting them. But the latter process would occupy much time, be very tedious and liable to error.

70. Let it be required to find the area of the floor of a rectangular room, the length of which is 8 yards, and breadth 4 yards. In other words, we are required to find how many times the *square yard* is contained in the floor.

Let fig. 101 represent a square yard, that is, the *unit* by which we measure areas; fig. 102 the floor, 6 yards long and 3 yards wide.

We first ascertain how many times the line AB is contained in A′B′, and find it to be 6 times. We then see how many times AD is contained in A′D′, and find it to be 3 times. Draw the straight line EF parallel to A′B′, and at a distance of *one* yard from the latter.

FIG. 101.

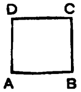
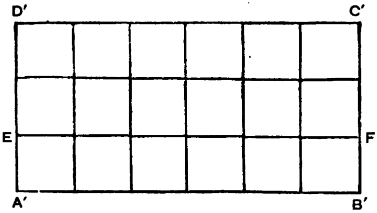


FIG. 102.

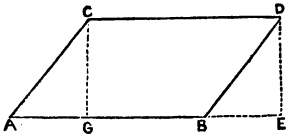


The rectangle A′B′FE evidently contains the square ABCD six times, that is, six times one square yard. In the whole surface we have three such rectangles as A′B′FE, and consequently 3 times 6 square yards, or 18 square yards is the area of the floor. In dividing the floor as shown in fig. 102, we have virtually multiplied the number of yards contained in its length by the number of yards contained in its breadth. Hence the rule for finding the area of a rectangle: *Multiply the length by the breadth.*

71. To find the area of a parallelogram ABDC (fig. 103).

Produce AB and draw the perpendiculars CG, DE. The

FIG. 103.



angles DBE, CAG, are equal as corresponding angles, CGA and DEB are equal being right angles; therefore ACG and BDE are also equal. The sides about these equal angles are also equal, therefore the two triangles ACG and BDE are equal. The parallelogram CDEG is rectangular, and it is composed of the right-angled trapezoid GCDB and the right-angled triangle BED. The parallelogram ABDC also consists of two figures, the trapezoid GCDB, and the *right-angled* triangle AGC. Thus

Rectangle CDEG = trapezoid GCDB + triangle BED
Parallelogram ABDC = trapezoid GCDB + triangle AGC

Rect. CDEG − parallelogram = triangle BED − triangle AGC.

But we have proved that the two triangles BED, AGC are equal; therefore we see that rectangle − parallelogram = 0; which shows that the rectangle and parallelogram are also equal.

72. We already know that by multiplying the length CD by the breadth CG we should find the area of the rectangle CDEG. The length CD of the rectangle is also the length of the parallelogram, and the breadth CG of the rectangle is the altitude or height of the parallelogram. Hence the area of the parallelogram is found by multiplying its length and, height together. This rule holds good for all parallelograms, that is, for every quadrilateral whose opposite sides are parallel. What we call the breadth in the rectangle is the altitude in the rhombus and parallelogram. In each case it is the perpendicular distance of the base from the side opposite.

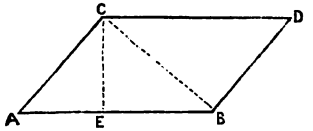
Since the square has all its sides equal, its area is obtained by squaring one of its sides, that is, multiplying the side by itself.

73. We have proved that the diagonal bisects a parallelogram, that is, divides it into two equal and similar triangles.

The area of the parallelogram ABDC = AB × CE; the triangle ABC being half the parallelogram ABDC, its area is equal to

$\frac{AB \times CE}{2}$ or $\frac{1}{2}(AB \times CE)$ or $\frac{AB}{2} \times CE$. Hence the rule for finding the area of a triangle: *Multiply half the base by the perpendicular height, or the base by half the perpendicular height.*

FIG. 104.

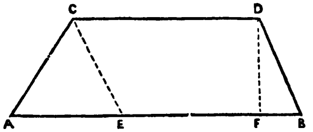


Hence it follows that triangles and parallelograms on the same base, or on equal bases, and between the same parallels, are equal and conversely.

74. To find the area of a trapezoid ABDC.

By drawing the line CE parallel to DB we divide the trapezoid into a parallelogram EBDC, and a triangle AEC.

FIG. 105.



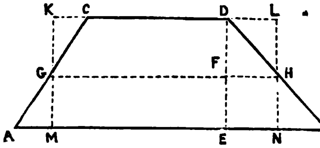
The area of the parallelogram $CDBE = EB \times DF$ (71). Area of triangle $ACE = \frac{1}{2} AE \times DF$. If these results be added together we obtain:

$$\text{Area } CDBE = EB \times DF.$$
$$\text{Area } ACE = \tfrac{1}{2} AE \times DF.$$
$$\text{Trapezoid } ABDC = (\tfrac{1}{2} AE + EB) \times DF = \tfrac{1}{2} DF\,(AE + 2\,EB) = \tfrac{1}{2}\,DF\,(AB + EB) = \tfrac{1}{2}\,DF\,(AB + CD)$$

Hence the following rule: To find the area of a trapezoid, *multiply the sum of the parallel sides by half the perpendicular distance between them.*

75. It is not necessary to know the two parallel sides of a trapezoid in order to find the area, as will appear from the following figure.

FIG. 106.



From the vertex D of the trapezoid ABDC draw the perpendicular DE. Bisect DE in F, and through the point F draw GH parallel to AB, and meeting the side AC and DB in G and H. Produce CD both ways to K and L, and through the points G and H draw KM and LN perpendiculars to AB, that is, parallel to DE. It is easy to prove that the triangles HBN and HDL are equal, and likewise GAM and GCK; therefore the trapezoid ABDC is equal to the rectangle MNLK=MN × NL. But MN=GH, and LN=DE. The line GH is termed the mean breadth of the trapezoid, and DE its altitude. Hence the following rule: To find the area of a trapezoid, *multiply its mean breadth by its perpendicular height or altitude.*

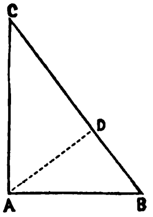
PART II.

CHAPTER VII.

THEOREM OF PYTHAGORAS.

76. From the vertex A of the right-angled triangle A B C, draw A D perpendicular to the hypothenuse B C. The triangle

FIG. 107.



A B C will be divided into two right-angled triangles A D B, A D C, which are similar to the right-angled triangle A B C. In the two triangles A B C, A D B, the angles C A B, A D B are equal, being right angles, and the angle B common to both, therefore the third angle C must be equal to D A B. Again in the two triangles, A B C, A D C, the angles B A C, A D C are equal, being right angles; the angle C is common to both triangles, therefore the third angle B must be equal to D A C. We see then that the angles of each of the triangles into which A B C is divided are equal respectively to the angles of A B C, hence each of the triangles A D B, A D C is similar to A B C.

The sides about the equal angles of similar triangles are proportional. Thus:—

$$DB : AB = AB : BC \qquad (1)$$
$$CD : AC = AC : BC \qquad (2)$$

These equalities may be written thus:

$$DB : AB :: AB : BC \qquad (3)$$
$$CD : AC :: AC : BC \qquad (4)$$

The latter two expressions (3) and (4) are termed proportions. The *first* and *fourth* terms are called *extremes*, and the second and third terms, *means*. The product of the *extremes* is always equal to the product of the *means*. Hence (3) and (4) may be written

$$(AB)^2 = DB \, . \, BC \qquad (5)$$
$$(AC)^2 = CD \, . \, BC \qquad (6)$$

If we add these two expressions together we have:

$$(AB)^2 + (AC)^2 = BC\,(DB + CD) \qquad (7)$$

or

$$(AB)^2 + (AC)^2 = BC \, . \, BC \qquad (8)$$
$$\therefore (AB)^2 + (AC)^2 = (BC)^2 \qquad (9)$$

From this expression we conclude that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the two sides which include the right angle.

This important proposition was discovered by Pythagoras. It is said that he first constructed a triangle, whose sides were in the ratio of 3, 4 and 5:

$$3^2 + 4^2 = 5^2$$
$$(2 \times 3)^2 + (2 \times 4)^2 = (2 \times 5)^2$$
$$(3 \times 3)^2 + (3 \times 4)^2 = (3 \times 5)^2$$
$$(4 \times 3)^2 + (4 \times 4)^2 = (4 \times 5)^2, \text{ \&c.}$$

Referring to equation (9) we may say that the square on the base of a right-angled triangle added to the square on the perpendicular is equal to the square on the hypothenuse.

Let us denote the *hypothenuse*, *base* and *perpendicular* of a right-angled triangle by *h*, *b*, and *p* respectively.

Hence equation (9) may be written

$$h^2=b^2+p^2$$

If any two of the three sides *h*, *b*, and *p* be given, the third can always be found from this equation. If *b* and *p* are known, then *h* is found from the following expression :

$$h=\sqrt{(b^2+p^2)}$$

That is to say, *h* is equal to the square root of the sum of the squares of *b* and *p*. If *h* and *p* are given, then

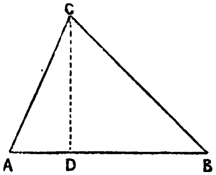
$$b=\sqrt{(h^2-p^2)}$$

If *h* and *b* are given, then

$$p=\sqrt{(h^2-b^2)}$$

77. The use of this theorem is not confined to right-angled triangles. By means of it we can express a side of any triangle in terms of the two sides which are known.

FIG. 108.



1°. Let it be required to express B C in terms of A C and A B (fig. 108). From the vertex C draw the perpendicular C D on the base A B.

$$BD=AB-AD.\therefore BD^2=(AB-AD)^2$$

$$CD^2=AC^2-AD^2, \text{ and } BC^2=CD^2+BD^2 \qquad (1)$$

Hence, if we substitute for CD^2 and BD^2 in (1) their equivalents, we obtain

$$\begin{aligned} BC^2 &= AC^2 - AD^2 + (AB - AD)^2 \\ &= AC^2 - AD^2 + AB^2 - 2\,AB.AD + AD^2 \\ &= AC^2 + AB^2 - 2\,AB.AD. \end{aligned}$$

In order to find an expression for AC,

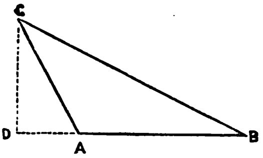
$$AD = AB - BD.\therefore AD^2 = (AB - DB)^2$$
$$CD^2 = BC^2 - BD^2, \text{ and } AC^2 = CD^2 + AD^2 \qquad (2)$$

Substituting these values for CD^2 and AD^2 in equation (2) we have

$$\begin{aligned} AC^2 &= BC^2 - BD^2 + AB^2 - 2\,AB.BD + BD^2 \\ &= BC^2 + AB^2 - 2\,AB.BD. \end{aligned}$$

78. In an obtuse-angled triangle ABC, the perpendicular from C is drawn on the production of BA. In this case

FIG. 109.



$$BD = AB + AD.\therefore BD^2 = (AB + AD)^2$$
$$CD^2 = AC^2 - AD^2, \text{ and } CB^2 = CD^2 + BD^2 \qquad (3)$$

Hence equation (3) may be written

$$\begin{aligned} CB^2 &= AC^2 - AD^2 + AB^2 + 2\,AB.AD + AD^2 \\ &= AC^2 + AB^2 + 2\,AB.AD. \end{aligned}$$

The expression for AC may be found by the pupil.

79. We shall now define what is meant by the *projection* of a line, and express Arts. 77, 78 in two general rules.

The *projection* of a point on a straight line is the foot of the perpendicular from the point on the line or the line produced. E and G are the projections of A and B, and the straight line E G is the projection of the straight line A B (fig. 110).

FIG. 110.

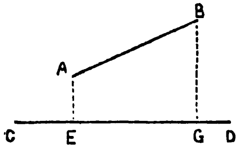
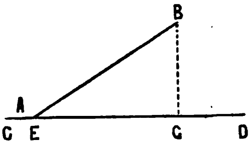


FIG. 111.



In fig. 111 the straight line A B meets C D, therefore the point A and its *projection* are coincident points—that is to say, they fall together; E G is still the projection of A B. When the line to be *projected* is parallel to the line on which it is to be projected, then the line and its projection are equal; when they are perpendicular to one another, the *projection* is a *point.* When they are inclined to one another, as in figs. 110, 111, the projection is less than the line projected. The greater the inclination the less the *projection.*

Rule 1.—In acute-angled triangles the square on the side opposite an acute angle is equal to the sum of the squares on the sides which contain it diminished by twice the product of either of these sides and the projection of the other upon it.

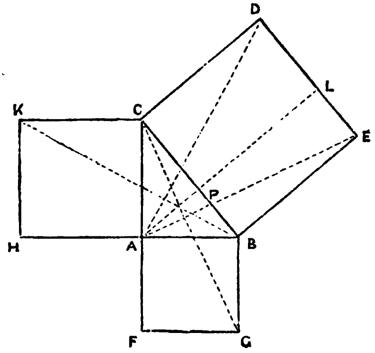
Rule 2. In obtuse-angled triangles, the square on the side opposite the obtuse angle is equal to the sum of the squares on the sides which contain it increased by twice the product of either of these sides and the projection of the other upon it.

80. We shall now give another demonstration of the theorem of Pythagoras.

On each of the sides of the right-angled triangle B A C construct squares B E D C, A H K C, and A B G F respectively.

Then C A F will be a straight line (43). From A draw A P perpendicular to B C and produce it to meet D E in L.

FIG. 112.



Draw the straight lines B K, A E, and C G. Then

B A = B G
B C = B E
A B E = C B G

∴ △ A B E is equal and similar to △ C B G

but the triangle A B E is equal to half the rectangle B E L P and the triangle C B G is equal to half the square A B G F, therefore the rectangle B E L P and the square A B G F are equal. Again

A C = C K
C D = C B
A C D = K C B

∴ △ A C D is = and similar to K C B.

But the triangle A C D is equal to half the rectangle C D L P, and the triangle B C K is equal to half the square A H K C, therefore the rectangle C D L P and the square A C K H are equal. Hence we see that the square on A B and the square on A C are together equal to the square on B C.

Since the rectangle C D L P is equal to the square A C K H, it follows that

$$CD \cdot CP = AC^2,$$

that is,

$$BC \cdot CP = AC^2 \qquad (1)$$

And because the rectangle B E L P is equal to the square A B G F

$$BE \cdot PB = AB^2$$

$$BC \cdot PB = AB^2 \qquad (2)$$

If we add (1) and (2) together, we obtain

$$BC\,(CP + PB) = AC^2 + AB^2$$

or

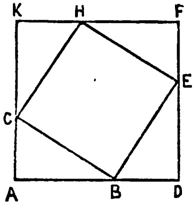
$$BC \cdot BC = AC^2 + AB^2$$

that is

$$BC^2 = AC^2 + AB^2$$

81. Another way of demonstrating the truth of this theorem is to describe a square C B E H on the hypothenuse

FIG. 113.



B C of a right-angled triangle A B C. Produce the sides A B and A C. Through the points E and H draw parallels to the productions of A C and A B respectively, and thus complete the square A D F K. Now the triangles B D E and A C B are equal and similar; for the angles B D E and C A B are equal, being right angles. The angle E B D is the complement of B E D, and also of C B A, therefore the angles B E D and C B A are equal. And because A C B, E B D are the complements of equal angles, they are also equal. Further, the hypothenuse B C of the triangle A B C is equal to the hypothenuse B E of the triangle D B E, for they are sides of the same square;

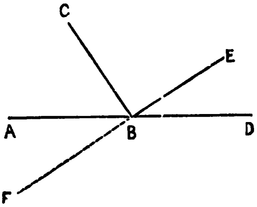
therefore the two triangles are equal. In a similar manner we can prove that the triangles KHC, FEH are each equal to ABC, and therefore the four right-angled triangles are equal and similar to one another.

Now, because $BD=AC$, and $AB=CK$, $BD+AB=AC+CK$ $\therefore AD=AK$, and therefore $AD^2=AK^2$, or $(AB+BD)^2=AK^2$ $=$square ADFK.

$$\begin{array}{lccc} & AB^2+2\,AB.BD+BD^2 & = & CBEH+4\,ACB \\ \text{But} & 2\,AB.BD & = & 4\,ACB \\ \hline & AB^2+BD^2 & = & BC^2 \end{array}$$

82. N.B. The angle EBD was said to be the complement of ABC. To show this, let EBC be a right angle. The angle

FIG. 114.



ABF=EBD, being vertically opposite angles, and ABF is the complement of ABC; hence EBD is also the complement of ABC.

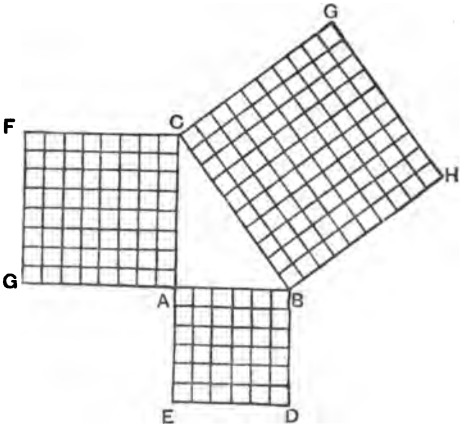
83. The truth of this theorem may be practically demonstrated in the following manner. Draw two lines BA, CA, meeting each other in the point A at right angles (fig. 115).

Take any *two numbers*, the sum of whose squares is a *perfect square*, and mark off on AB, AC as many equal parts as these numbers indicate respectively. For example, divide AB into six equal parts and AC into eight equal parts, draw BC and divide it into ten equal parts. Complete the squares ABDE, ACFG, BCGH, as shown in the figure. The number of squares on the hypothenuse BC will be found equal to the sum of those on the sides A B and A C of the right angle.

$$6^2+8^2=10^2$$

This may be called a particular example of a theorem which is generally true.

FIG. 115.



Diagonals.

84. Any straight line which joins two non-consecutive adjacent vertices of a polygon is termed a *diagonal.* In a quadrilateral only *one* diagonal can be drawn from the same vertex (fig. 116).

FIG. 116.

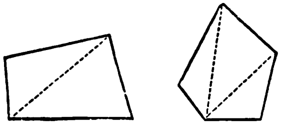
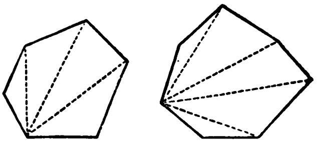


FIG. 117.



In a five-sided figure, or pentagon, *two* diagonals can be drawn from the same vertex.

In a six-sided figure, or hexagon, *three.*

In a seven-sided figure, or heptagon, *four.*

85. Thus we find that the number of diagonals which can be drawn from the same vertex of a polygon is always equal to the number of sides of the polygon diminished by *three.* In the quadrilateral *one* diagonal divides the figure into *two* triangles; in the five-sided figure *two* diagonals divide the figure into *three* triangles; in a hexagon, three diagonals divide it into *four* triangles; and in the seven-sided figure, four diagonals divide it into *five* triangles. It appears, then, that the number of triangles into which a polygon can be divided, by drawing all the diagonals possible from the same vertex, is always equal to the number of diagonals increased by one.

86. Again, we find that the *four-sided* figure, or quadrilateral, is divided into *two* triangles; the *five-sided* figure, or pentagon, into *three*; the *six-sided*, or hexagon, into *four*; the *seven-sided,* or heptagon, into *five,* and so forth. From this we see that the number of triangles thus formed is invariably equal to the number of sides of the polygon *less two.* If a polygon have n sides, $n-3$ diagonals can be drawn from the same vertex, and from what has just been stated, $n-3+1$ or $n-2$ triangles will be formed.

87. As we know the number of triangles into which any polygon can be divided in the way just indicated, it is easy to find the number of right angles to which all the angles of a polygon are equal; and consequently the number of degrees in all the angles of that polygon.

Name of Polygon	No. of sides and No. of Angles	No. of Triangles	No. of Right Angles	No. of Degrees
Quadrilateral	4	2	2 × 2	360°
Pentagon	5	3	3 × 2	540°
Hexagon	6	4	4 × 2	720°
Heptagon	7	5	5 × 2	900°
Octagon	8	6	6 × 2	1080°
Nonagon	9	7	7 × 2	1260°
Decagon	10	8	8 × 2	1440°

88. If we suppose the figures to be *regular polygons*—that is, polygons whose sides and angles are equal—the magnitude of each angle may be found by dividing the total number of degrees in the polygon by the number of its angles; thus, in a

Quadrilateral	$360^\circ \div 4 = 90^\circ$	in each angle
Pentagon	$540^\circ \div 5 = 108^\circ$	"
Hexagon	$720^\circ \div 6 = 120^\circ$	"
Heptagon	$900^\circ \div 7 = 128\frac{4}{7}^\circ$	"
Octagon	$1080^\circ \div 8 = 135^\circ$	"
Nonagon	$1260^\circ \div 9 = 140^\circ$	"
Decagon	$1440^\circ \div 10 = 144^\circ$	"

89. The greatest number of diagonals which can be drawn in any polygon is easily ascertained. In a nine-sided polygon or nonagon, $9-3=6$ diagonals can be drawn from the same vertex, and since the polygon has nine vertices, it would appear that $9 \times 6 = 54$ diagonals could be drawn. Now, each diagonal unites two vertices, therefore we shall not have 54 distinct diagonals, but $54 \div 2 = 27$. Hence in a nonagon 27 diagonals can be drawn.

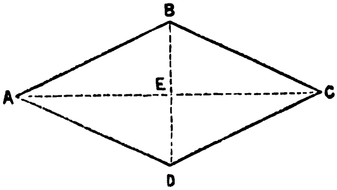
90. Generally; in a polygon of n sides, $(n-3)$ diagonals can be drawn from one vertex, and $n\,(n-3)$ from all the vertices. But each of the $n\,(n-3)$ diagonals has been taken twice, therefore the number of distinct diagonals which can be drawn in a polygon of n sides is $\frac{n\,(n-3)}{2}$.

CHAPTER VIII.

POLYGONS.

91. THE diagonal A C of the rhombus A B C D divides it into two isosceles triangles A B C and A D C.

FIG. 118.



By drawing the second diagonal B D, the rhombus is divided into four right-angled triangles; for the angle B A E =B C E, and A B E=C B E; therefore the angle A E B=C E B. But A E B=C E D=C E B=A E D=90°.

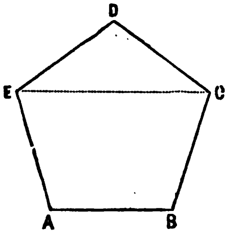
$$\text{The area of } ABC=\tfrac{1}{2}\,(AC\,.\,BE)$$
$$\text{The area of } ADC=\tfrac{1}{2}\,(AC\,.\,DE)$$
$$\text{The area of rhombus}=\tfrac{1}{2}\,AC\,(BE+DE)=\frac{AC\,.\,BD}{2}$$

Hence we see that the area of a rhombus is found by taking half the product of its two diagonals.

92. By drawing diagonals in regular polygons the latter will be divided into figures whose areas are easily computed. Let us take a regular *pentagon* A B C D E as an example (fig. 119).

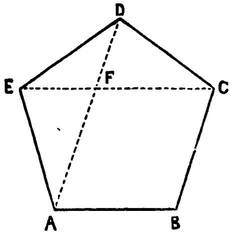
The diagonal E C divides the polygon into an isosceles triangle E C D, and an isosceles trapezoid A B C E. We know that the angle E D C=108°.·.D E C+D C E=180°−108°=72°. And because D E C and D C E are equal, each angle is equal to 36°. The magnitude of each of the equal angles A E C, B C E is 108°−36°=72°. Because E A B is equal to 108° and A E C is equal to 72°, it follows that E A B and A E C are supplementary angles, for their sum is equal to two right angles or 180°; therefore the straight lines E C and A B are parallel (Art. 47). And the sides A E, B C are equal, being sides of a regular polygon; consequently A B C E is an isosceles trapezoid.

FIG. 119.



93. If two diagonals E C, A D be drawn, the polygon is divided into *three* isosceles triangles, and *one* rhombus.

FIG. 120.

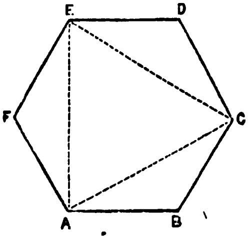


We have just proved that the angle E A D=36° and A E C =72°.·.A F E=180°−(36°+72°)=72°. Therefore the angles A E F, A F E are equal, consequently A E=A F; hence A E F is an isosceles triangle. In a similar manner we can prove that C D F is an isosceles triangle. The two triangles D E C, E A D are equal, having their vertical angles and angles at the base respectively equal, and the sides about the equal angles equal, therefore E C − F C = A D − A F, whence F D = F E, and consequently F D E is an isosceles triangle. We have thus shown that the sides of the figure A B C F are all equal, and by Art. 92 the opposite sides are parallel, therefore the figure is a rhombus.

In the same manner it can easily be shown that by drawing the diagonal A C five isosceles triangles are formed. And by drawing the diagonal E B, five isosceles triangles and an isosceles trapezoid are obtained.

94. The diagonals which can be drawn in a regular hexagon (fig. 121), are not all of the same length. They may therefore, for the sake of convenience, be distinguished as diagonals of the *first order* for the shorter, and diagonals of the *second order* for the longer.

FIG. 121.

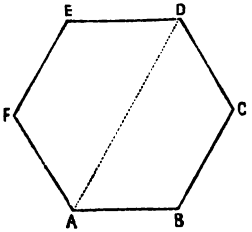


1°. The diagonals of the first order A C, A E, C E, divide the figure into three isosceles triangles and one equilateral triangle. For the side A B = B C = C D = D E = E F = F A, and the angle A B C = C D E = E F A = (4 × 180°) ÷ 6 = 120°. The triangles A B C, C D E, E F A are equal and similar, consequently the bases A C, C E, E A are equal, and therefore the triangle A C E is equilateral.

2°. The two diagonals of the first order A E, B D divide the figure into two isosceles triangles and a rectangle. It has been already shown that A F E is an isosceles triangle, and that the angle E F A=120°, each of the angles F A E, F E A =(180° − 120°)÷2=30°. Hence the angle A E D is equal to (120° − 30°)=90°, and is therefore a right angle. This is sufficient to show that the figure A E D B is a rectangle.

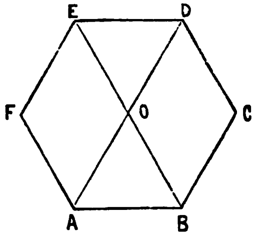
3°. By drawing a diagonal A D of the second order, the hexagon is divided into two isosceles trapezoids. The two angles B A F, C D E are bisected by the diagonal A D. Hence each of the angles B A D, C D A, F A D, E D A is equal to 120°÷2=60°. Each of the equal angles D C B, C B A, D E F, E F A is the supplement of each of the equal angles C D A, B A D, E D A, F A D; consequently the sides C B and E F are parallel to A D, and because the sides of the hexagon are all equal by hypothesis, the two figures A B C D, A F E D are isosceles trapezoids.

FIG. 122.



4°. If two diagonals A D, B E of the second order be drawn, the hexagon is divided into two equilateral triangles

FIG. 123.

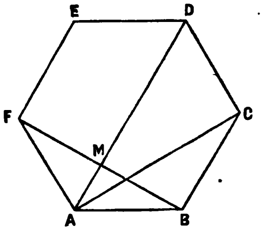


and two rhombuses. We know (3°) that each of the angles O E D, O D E is equal to 60°, therefore the remaining angle EOD is also equal to 60°, consequently O E D is an equilateral triangle, and OE=ED=EF. In like manner we can prove that AO=AB=AF=FE. Now, because the opposite sides of the figures O B C D, O A F E respectively are both equal and parallel, these figures are rhombuses.

5°. If the radii O F, O C be drawn, then F O, O C will be in the same straight line, and therefore F C will be a diagonal. For each of the angles F O E, E O D, D O C is one-third of two right angles; therefore their sum is equal to two right angles; hence F O, O C are in the same straight line, and the figure will be divided into six equal equilateral triangles.

6°. If from the same vertex A two diagonals A C, A D of the first and second order respectively be drawn, which do not intersect, the hexagon is divided into three figures which are all different, viz. an isosceles triangle A B C, a right-angled triangle A C D, and an isosceles trapezoid A F E D.

FIG. 124.



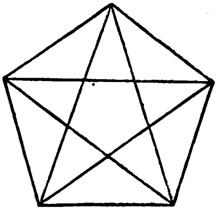
7°. If two diagonals, B F of the first order and A D of the second order, be drawn, which do intersect, the hexagon is divided into four figures, viz. two right-angled triangles A M B, A M F, and two right-angled trapezoids M B C D and M F E D.

CHAPTER IX.

INTERIOR POLYGONS.

95. In the case of a regular pentagon, diagonals of one order only can be drawn (fig. 125). By drawing all the diagonals possible a smaller pentagon is formed whose vertices are their intersections. The position of the interior pentagon is the reverse of that of the exterior, and its sides are parallel respectively to the corresponding sides of the latter.

FIG. 125.

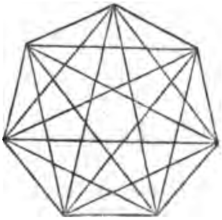


96. In the regular hexagon two kinds of diagonals can be drawn. Those of the first order form an interior hexagon similar to the larger polygon. The diagonals of the second order intersect in the same point within the figure, and therefore do not form a new figure.

97. In the regular heptagon the diagonals of the *first order* form an interior heptagon. The position of the latter is the reverse of that of the larger heptagon. If the smaller heptagon be turned upside down, its sides will be parallel to

the corresponding sides of the larger polygon. The two polygons are similar to one another. If the diagonals of the

FIG. 126.



second order be drawn, a heptagon will be formed, the position of which is the same as that of the exterior polygon. Diagonals of the *third order*—that is, which pass over three vertices—cannot be drawn in a heptagon.

98. In the case of a regular octagon, the diagonals of the *first order* form an interior octagon, the position of which is the reverse of that of the larger octagon. The diagonals of the *second order* form an interior octagon, the position of which is the same as that of the larger octagon. The diagonals of the *third order* intersect in the *same point*, and therefore do not form a new figure.

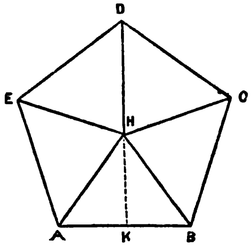
99. In a regular nonagon, diagonals of the *first*, *second*, and *third order* will form interior nonagons. The position of the nonagons formed by the diagonals of the first and third order respectively is the reverse of the position of the large polygon. The position of the interior polygon formed by diagonals of the second order is the same as that of the large polygon.

100. It has been stated (Art. 86) that $n-3$ diagonals from any vertex can be drawn in every polygon. The number of interior polygons which can be formed will always be found to be equal to $\frac{n-3}{2}$ when the number of sides of the polygon is odd—that is, when n is odd; and to $\frac{n-4}{2}$ when it is even. Moreover, the diagonals of the first, third, fifth, &c., order always form the interior polygon in a reversed position to

that of the large polygon respectively, whilst those of the second, fourth, &c., form the interior polygon in the same position as that of the external polygon unless they all pass through the same point.

101. In order to find the area of a regular polygon A B C D E (fig. 127), bisect two consecutive angles A and B.

FIG. 127.



From the point H where the two bisectors of the angles A and B cut one another draw lines to the vertices E, C, D respectively. The polygon will be divided into as many triangles as the figure has sides. These triangles are all equal and similar, for A E=A B and H A is common, therefore H E=H B. Hence the triangles A H E, A H B are equal. By drawing the perpendicular H K we can find the area of the triangle A H B, and as all the triangles are equal, the area of the polygon is expressed by

$$5\,AB \times \frac{HK}{2}$$

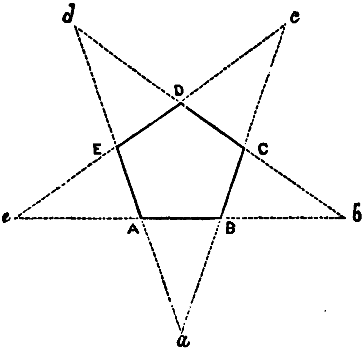
Since A E=E D=D C=C B=B A, 5 A B is the sum of the sides of the polygon. The sum of the sides of a polygon is called the *perimeter*. The point H is the *centre* of the polygon, and a perpendicular from this point on a side is termed the *apotome* of the polygon. From the formula $5\,AB \times \frac{HK}{2}$ we see that the area of a regular polygon is measured by the *product of the perimeter and half the apotome.*

CHAPTER X.

STAR-POLYGONS, EQUIVALENT FIGURES, &c.

102. If the sides of a regular quadrilateral be produced they will never meet, and consequently no new figure can be formed in this manner. If, however, the sides of a regular pentagon be produced in both directions, five distinct intersections will be obtained, which are the vertices of a new pentagon.

Fig. 128.

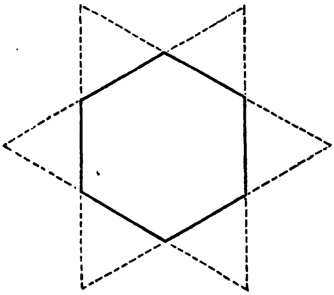


In the regular pentagon A B C D E, if the sides be produced, the intersections *a*, *e*, *d*, *c*, *b*, are formed. The triangles A *e* E, E *d* D, D *c* C, C *b* B, and B *a* A are all equal and similar; for the angle $AED = 108° \therefore dED = 180° - 108 = 72°$, so also $dDE = 72°$; hence the triangle *d*D E is an isosceles triangle. In the same way it can be shown that each of the angles at the base of each of the isosceles triangles is equal to

72°. Thus D d=E d=E e=A e, &c. By drawing the straight lines $a\,e$, $e\,d$, $d\,c$, $c\,b$, $b\,a$ a regular polygon similar to ABCDE is formed.

The figure formed by producing the sides of a polygon is termed a *star-polygon.*

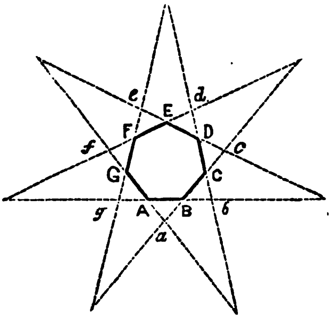
FIG. 129.



103. By producing the sides of a hexagon in both directions, a star-polygon of 12 sides is formed (fig. 129).

In fig. 130 we have a heptagon. If the sides be produced they will intersect in the seven points a, b, c, d, e, f, g respectively; and by producing the sides beyond these points, seven new points of intersection will be formed and a second star-polygon will be obtained.

FIG. 130.



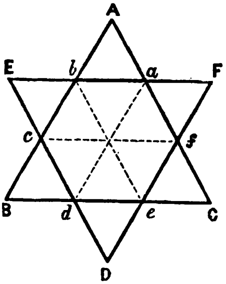
104. In like manner if the sides of a regular octagon be produced, *two star-polygons* will be formed. By producing the sides of a nonagon we shall obtain *three star-polygons.*

We have seen that by producing the sides of a regular polygon of n sides, $\frac{n-4}{2}$ star-polygons can be formed, when n is an even number; and $\frac{n-3}{2}$ star-polygons when n is odd.

Intersection of Figures.

105. Of two equal triangles one may be placed on the other in such a manner that the two triangles do not coincide—that is, exactly cover one another. For instance,

FIG. 131.



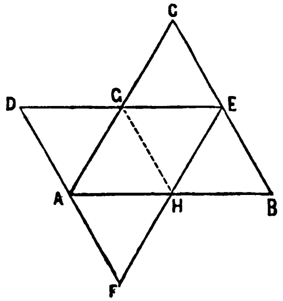
they can be placed together so as to form 1, 2, 3, 4, 5, 6 points of intersection. In the case of two equal equilateral triangles one can be placed on the other so as to form (1) six equal equilateral triangles, (2) a regular star-polygon, (3) a regular hexagon equal in area to the sum of the six equal equilateral triangles. In order to obtain these results, each side of one of the equilateral triangles is trisected—that is, divided into three equal parts as in the figure.

106. Of two equal isosceles triangles one can be placed on the other so as to form six equal isosceles triangles, the sum of whose areas is equal to the area of the interior six-sided polygon. The latter is not a regular polygon in this case.

The two triangles must be placed together as indicated in fig. 131. To prove that the triangles *a b* A, *c b* E, *c d* B, *e d* D, *e f* C, and *a f* F are isosceles, it is only necessary to observe that *e d* is parallel to E F, *e f* to A B, *a f* to D E, *a b* to C B, *c b* to D F, and *d c* to A C; the triangles *e* D *d*, *f* F A, and *e* E *b* are similar to the triangle C A B and the other three to E D F; hence these triangles are isosceles.

107. Of two equal equilateral triangles, one can be placed on the other so as to form four equal equilateral triangles and a rhombus. Half the sum of these triangles (fig. 132) is

FIG. 132.

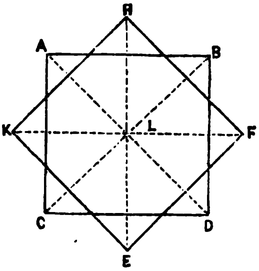


equal to the area of the quadrilateral A H E G. Thus, bisect the sides of the triangle A B C in E, G, H. Draw the straight lines E D, E F, F A through the points G, H, and A respectively, making them equal to the sides of the triangle A B C, then the triangle D F E will be equal and similar to the triangle A B C; draw G H and we obtain six equilateral triangles. The other truths are obvious.

108. The intersections formed by placing one quadrilateral upon another are worth notice. Two equal squares, for instance, can intersect one another in not more than eight points, and in such a manner that eight right-angled isosceles triangles are formed.

In the square A B D C draw the diagonals A D, B C. Through their point of intersection L draw L K, L F parallel to A B, and L F, L E parallel to A C, making each line equal to A L. Join H K, H F, F E, E K. The quadrilateral E F H K is a square, because the diagonals K F and H E are equal and cut one another at right angles. The square E F H K is equal to the square A B D C, because

FIG. 133.

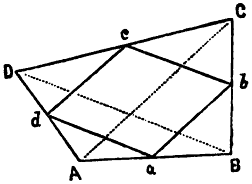


$$KH^2 = HF^2 = FE^2 = EK^2 = HL^2 + KL^2 = AL^2 + LB^2 = AB^2 = BD^2.$$

109. If two quadrilaterals meet in four points, so that one falls entirely within the other, the smaller figure is said to be inscribed in the larger. As particular cases the two following may be mentioned :

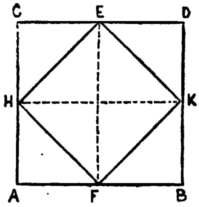
1°. Bisect the sides of the quadrilateral A B C D in points *a*, *b*, *c*, *d* respectively. The quadrilateral *a b c d* thus formed is, 1°, a parallelogram. 2°. Its area is equal to half the area of the quadrilateral A B C D, or, what amounts to the same thing, the area of the parallelogram is equal to the sum of the four triangles *c d* D, *c b* C, *a b* B, *a d* A.

FIG. 134.



The square A B D C is divided into four equal parts by drawing H K, E F through the points of bisection E, K, F, H of the sides of the square. The figure H E K F is an inscribed square, and is equal to half the area of A B D C. It is also

FIG. 135.



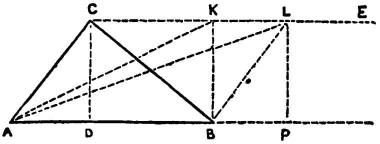
equal to the sum of the four triangles E K D, K F B, H F A, and E H C. These are right-angled isosceles triangles, equal and similar to one another; the area of each being equal to one-eighth of that of the square A B D C.

Equivalent Figures.

110. To convert one figure into another of equal area.

In our demonstrations of the areas of figures it was shown how the area of a parallelogram was equal to that of a rectangle, &c. (§ 71). In the present article we shall show how

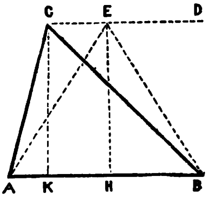
FIG. 136.



to change one figure into another without altering its area. Triangles and quadrilaterals only will be spoken of.

Let A B C (fig. 136) be an acute-angled triangle. The area of this triangle is $=\frac{1}{2}$ A B $\times$ C D. Through the point C draw a line parallel to A B and produce A B. If from any point in C E lines be drawn to A and B, a triangle will be formed equal in area to the triangle A B C. If the point in C E be so taken that the perpendicular from it on A B passes through B, the triangle formed is a right-angled triangle. If the perpendicular fall on A B produced, the triangle formed is obtuse-angled. In fig. 136 the area of $AKB = \frac{1}{2} AB . KB$, and since $KB = CD$, the area of the right-angled triangle $= \frac{1}{2} AB . CD$ $=$ area A C B. Again, the area of $ALB = \frac{1}{2} AB \times LP$, but $LP = CD$, therefore the area of the obtuse-angled triangle $ALB = \frac{1}{2} AB . CD$.

FIG. 137.



111. The acute-angled triangle A B C can also be reduced to an isosceles triangle of equal area. Through the point C draw a line C D parallel to A B; bisect A B in H and draw H E perpendicular to A B, meeting the parallel through C in E. Join E A and E B and the isosceles triangle A E B shall be equal in area to the acute-angled triangle A B C. For

$$\text{Area of } ABC = \tfrac{1}{2} AB . CK$$
$$\text{Area of } AEB = \tfrac{1}{2} AB . EH.$$

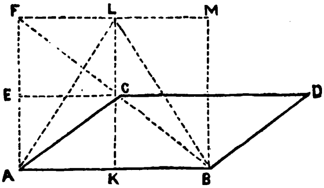
But $CK = EH$, therefore $\frac{1}{2} AB . CK = \frac{1}{2} AB . EH$, or the area of A B C is equal to the area of A E B.

112. In a similar manner an obtuse-angled triangle can be changed into an acute-angled triangle, a right-angled triangle, or an isosceles triangle. An isosceles triangle can be changed into an obtuse-angled triangle, a right-angled triangle, or an acute-angled triangle. An equilateral triangle can likewise be converted into an equivalent acute, obtuse, or right-angled triangle.

To change an obtuse-angled triangle into an equilateral triangle it is necessary to know certain properties of the circle.

113. To construct a triangle equivalent to a given parallelogram.

FIG. 138.



1°. A right-angled triangle.

The area of a parallelogram =length multiplied by altitude. At the point A draw A F perpendicular to A B. Produce D C to meet A F in E. The area of the parallelogram A B D C=A B . A E. Make A F=2 A E; join the points F and B. B A F is a right-angled triangle.

Area of A B D C=A B . A E

$$=\tfrac{1}{2} AB \times 2 AE$$

$$=\tfrac{1}{2} AB \times AF = \text{the right-angled triangle } ABF.$$

2°. An isosceles triangle.

The same construction being used. Through F draw F M parallel to A B. Bisect A B in the point K, and draw K L perpendicular to A B, and meeting F M in L. Join the points L and A and L and B. A B L is the isosceles triangle required.

$$\text{Area of } ABL=\tfrac{1}{2} AB \times KL$$

$$=\tfrac{1}{2} AB \times 2 AE$$

$$= AB \times AE.$$

Hence we see that the area of the isosceles triangle A B L is equal to the area of the parallelogram A B D C.

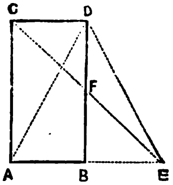
3°. An acute-angled triangle.

Draw F M parallel to A B. Draw B M perpendicular to A B, and meeting F M in M. By joining any point between F and M to A and B we obtain an acute-angled triangle at A or B equivalent in area to the parallelogram. If any point beyond M or F be joined to A and B, an obtuse-angled triangle will be formed.[1]

114. To change a rectangle into a right-angled triangle or an isosceles triangle of equal area.

In the rectangle A B D C produce A B to E, making B E equal to A B. Join the points C and E.

FIG. 139.

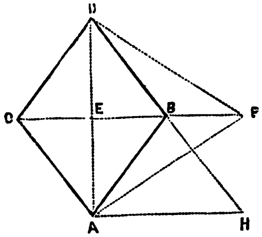


Because by § 57 the triangle B F E is equal and similar to the triangle D F C, the area of A B D C is equal to the area of A C E.

From the point D draw D A, D E. The isosceles triangle A D E is equal in area to the rectangle A B D C.

115. To change a rhombus or a square into a right-angled or an isosceles triangle of equal area.

FIG. 140.



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[1] If A B is less than A C, such a triangle can always be found; but if A B is greater than A C, then the construction cannot always be effected.

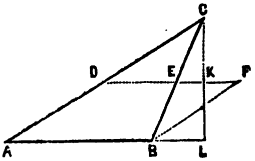
Draw the diagonals AD, BC. From the point A draw a perpendicular to AD, meeting DB produced in H. The area of a rhombus is equal to half the product of its two diagonals (Art. 91).

Area of $ABDC=\frac{1}{2}AD\times CB$. Area of the right-angled triangle $AHD=\frac{1}{2}AD\times AH$, or, since $AH=CB$, the area of $AHD=\frac{1}{2}AD.CB$. If we produce CB to F, making $BF=\frac{1}{2}BC$, and join F to A and D, the area of the isosceles triangle AFD is equal to the area of the rhombus $ABDC$. For the area of AFD is$=\frac{1}{2}AD.EF=AD\times\frac{1}{2}EF=AD\times CB=$area of the rhombus $ABDC$.

In a similar manner it may be shown that a square can be changed into a right-angled triangle or an isosceles triangle.

116. To change a triangle into a parallelogram.

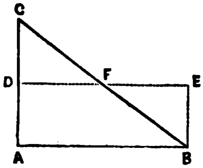
FIG. 141.



Bisect one of the sides AC of the triangle ABC in the point D; draw DE parallel to AB and BF parallel to AD, meeting DE produced in F. The parallelogram $ABFD$ shall be equal to the triangle ABC. To prove this it is only necessary to remark that the triangles DEC, BEF are equal and similar, having the vertical angles at E equal, and the alternate interior angles at F and D, and likewise those at C and B respectively, equal, and the side DC equal to BF, therefore the two triangles are equal. Moreover, the height of the parallelogram is equal to half the height of the triangle; for AB and DF being parallel, the perpendicular CL is, like AC, bisected by DF. Thus the area of $ABC=\frac{1}{2}AB\times CL=\frac{1}{2}AB\times 2KL=AB\times KL=$area $ABFD$.

117. To change a right-angled triangle A B C into a rectangle A B E D.

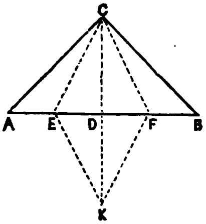
FIG. 142.



Bisect the hypothenuse C B in F. Through F draw D E parallel to A B, and through the point B draw B E parallel to A C. The rectangle A B E D is equal in area to the right-angled triangle A B C. For the triangles B F E and F D C are equal and similar (115).

118. To change an isosceles triangle into a rhombus.

FIG. 143.



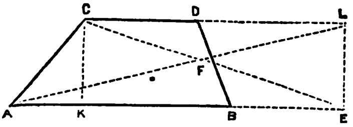
Let A B C be an isosceles triangle. From the vertex C let fall the perpendicular C D on A B and produce it to K, making D K equal to C D. Bisect A D, D B in points E and F respectively, and draw E C, C F, F K and K E. The figure E C F K is a rhombus whose area is equal to that of the isosceles triangle A B C. The four right-angled triangles E C D, F C D, F K D, E K D are equal and similar, hence the hypothenuses E C, C F, F K, E K are all equal, and consequently the figure E C F K is a rhombus. Because the triangles A E C, E D C have equal bases and the same altitude, and $AEC = \frac{1}{4} ABC$, we have

$4EDC = 4AEC$, or the rhombus ECFK equal to the isosceles triangle.

From this it is easy to see how a square, rectangle, and parallelogram can be changed, first into an isosceles triangle and then into a rhombus.

119. To change a trapezoid into a triangle. Let ABDC be a trapezoid. Produce AB to E, making BE equal to CD.

FIG. 144.

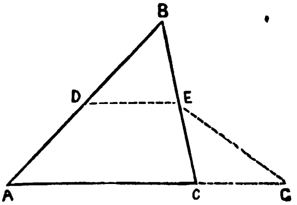


Join CE cutting BD in F. The triangle ACE is equal in area to the trapezoid ABDC. It is easy to prove that the triangle BFE is equal and similar to the triangle DFC. We know that the area of the trapezoid $ABDC = \frac{1}{2}CK(CD + AB) = \frac{1}{2}CK(BE + AB) = \frac{1}{2}CK \times AE =$ area of triangle AEC. If a right-angled triangle were required, we should draw a perpendicular from E meeting CD produced in L, and then draw the straight line AL. AEL would be the right-angled triangle required.

The case of an isosceles triangle is left as an exercise for the pupil.

120. To change any triangle into a trapezoid. Let ABC

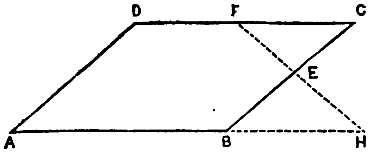
FIG. 145.



be a triangle. It is required to construct a trapezoid equal to it in area. Bisect either AB or CB, and through the point of bisection draw D E parallel to the base A C. Produce A C, making C G equal to D F; join E G. The trapezoid D E G A is equal in area to the triangle A B C. For the two triangles D E B, G C E, have equal bases and equal altitudes, consequently their areas are equal.

121. To change a parallelogram into a trapezoid.

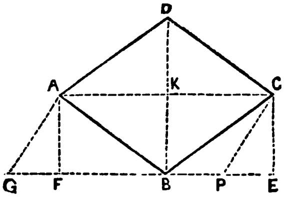
FIG. 146.



Bisect B C, and through the point of bisection draw a straight line meeting C D in the point F and A B produced in H. The trapezoid A H F D is equal in area to the parallelogram A B C D. If a right-angled trapezoid be required, then F H must be drawn through E perpendicular to D C, or to A B produced. If an isosceles trapezoid is to be constructed, we must make the angle B E F equal to twice the angle E C F, and produce F E to meet A B produced in H. Then the figure A H F D is the isosceles trapezoid required.

122. To change a rhombus into a rectangle and a parallelogram.

FIG. 147.

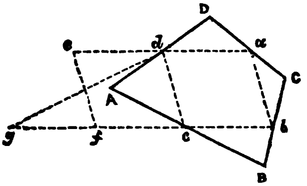


Let A B C D be a rhombus. Draw the diagonal A C, and through B draw E G parallel to A C, from A and C let fall perpendiculars A F, C E on G E. The rectangle A C E F will be equal to the rhombus ABCD. The diagonals AC, BD bisect one another at right-angles, the four triangles CKB, CKD, AKB, AKD are thus equal and similar to each other, and as the triangle CEB=KBC, and AFB=AKB, the equality of the two areas ACEF, ABCD is manifest. But we also know that the area of the rhombus$=\frac{1}{2}\text{AC}\times\text{BD}=\text{AC}\times\frac{1}{2}\text{BD}=\text{AC}\times\text{KB}=\text{AC}\times\text{AF}=$area of rectangle ACEF.

To change the rhombus into an equivalent parallelogram, we must draw through the point B a parallel to AC, and join C to any point P in EG (except the point E) and draw AG parallel to CP, the figure ACPG will be a parallelogram.

123. To change an irregular quadrilateral into a parallelogram or a trapezoid.

FIG. 148.

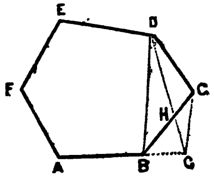


Let ABCD be an irregular quadrilateral. Bisect each side, and join *a b*, *b c*, *c d*, *d a*. The inscribed figure *a b c d* is equal in area to half the quadrilateral ABCD (109). Produce *b c* indefinitely, and make $cf=bc$, produce *a d* to *e*, making $de=ad$, join *ef*. The parallelogram *a b f e* is equal in area to the quadrilateral ABCD. If *fg* be made equal to *f c*, and the points *d g* joined, the trapezoid *a d g b* thus formed will also be equal in area to the quadrilateral ABCD.

124. To change an irregular polygon into another irregular polygon having one side less than the first, this second polygon into another of one side less, and so on, until we have changed the first polygon into an equivalent triangle.

Let A B C D E F be an irregular polygon. Draw a diagonal B D of our first order. Produce A B and draw C G parallel to B D, and meeting A B produced in G. Join the points D and G,

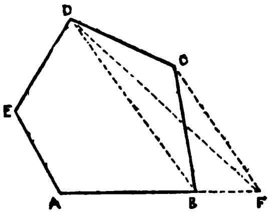
FIG. 149.



and the irregular six-sided figure A B C D E F will be reduced to an equivalent figure of five sides. For the triangles B D G and B C D are equal, therefore B D G—B H D=B C D—B H D, or B H G=D H C.

125. To change this five-sided figure into a quadrilateral.

FIG. 150.

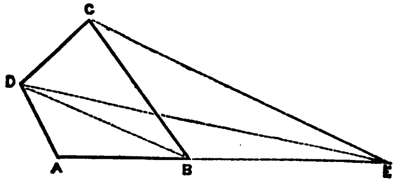


Begin by drawing a diagonal of the first order, B D, and through C draw C F parallel to B D and meeting A B produced in F. Join the points D and F, the figure A E D F shall be equal in area to the irregular pentagon A B C D E.

126. To change this quadrilateral into an equivalent triangle.

Again commence by drawing a diagonal B D, make C E parallel to B D, meeting the production of A B in E. Join the points D and E. The triangle A E D is equal in area to the quadrilateral A B C D. Thus we see that the area

FIG. 151.



of the triangle A E D = area A B C D = area A B C D E = area A B C D E F.

Circle.

127. The space enclosed by a curved line, which returns to its point of departure, and every point of which is equally distant from a certain point within the space, is called a *circle*.

The curved line itself is termed the circumference of the circle, the space enclosed is the *circle-plane*, and the point within the circle from which every point of the circumference is equally distant is the *centre* of the circle.

A circumference may be more briefly defined as the *locus or path of points equally distant from a fixed point.* We may then define the circle as the *space enclosed by this circumference* and the fixed point as *the centre of the circle.*

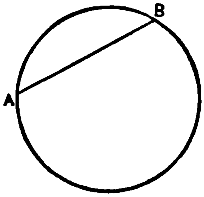
128. Any straight line drawn from the centre to the circumference is a *radius.* Two or more such straight lines are termed *radii.* It follows from the definition of a circumference that all radii of the same circle are equal.

129. Any portion of a circumference, however great or however small, is termed an *arc.*

The Circle and Straight Line.

130. A circle is said to be intersected by a straight line when the latter cuts the circumference in two points A B, fig. 152.

FIG. 152.



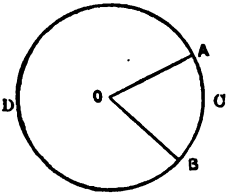
When such a line passes through the centre of the circle, it is called a *diameter*.

131. The straight line A B is called a *chord*, or more particularly, the *chord of the arc* A B.

A chord divides the circle into two parts, each of which is called a *segment* of the circle. Thus a segment of a circle is contained by an arc and its chord.

132. If two radii O A, O B, fig. 153, be drawn, the circle is divided into two parts which are called *sectors* of the circle.

FIG. 153.



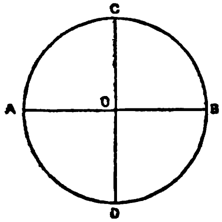
Thus a *sector* is contained by an arc and two radii. Or we may define a sector as a portion of the area of a circle bounded by two radii and the arc which they intercept.

A sector of a circle always consists of a segment plus or minus an isosceles triangle. Thus in fig. 153:

Sector O A C B = Segment A C B + triangle A O B,
and Sector O A D B = Segment A D B − triangle A O B.

133. When two diameters A B, C D, fig. 154, cut one another at right angles, the circle is divided into four equal parts, each of which is called a *quadrant.*

FIG. 154.



The right angle C O B is formed by the two radii O B, O C, and the arc B C between their extremities is one-fourth of the whole circumference.

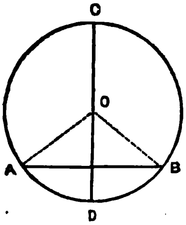
134. We have already seen (6) that by an inward or outward movement of either of the legs O B, O C, the angle B O C will become smaller or larger, and so also will the arc B C. And since the magnitude of an angle is independent of the length of its sides, it is manifest that its magnitude can be determined by the length of the arc to which it corresponds. The circle is divided into 360°, the arc A B is one-fourth of the whole circumference, and as it corresponds to a right angle, we may say that the angle C O B is measured by the arc B C, that is, by one-fourth of $360^\circ = 90^\circ$. Thus, *an angle at the centre of a circle is measured by the arc to which it corresponds.*

A fifth part of the circumference will correspond to an angle$=\frac{360^\circ}{5}=72^\circ$, &c.

135. The two lines which intersect at right angles do not always pass through the centre.

In fig. 155 the diameter C D cuts the chord A B, which does not pass through the centre, at right angles. By drawing radii O A, O B, an isosceles triangle O A B is formed, O A being equal to O B. The chord A B is bisected by C D (55). Hence follows the equality of the angles D O A, D O B (55); and consequently the equality of the arcs A D, D B (134). From this we see that arc C A D—arc A D=arc C B D—arc D B, or arc C A=arc C B.

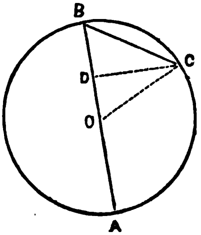
Fig. 155.



136. We shall now consider the case when a diameter and chord meet at a point on the circumference.

In fig. 156 let a diameter A B and chord C B meet at the point B. The angle A B C thus formed is measured by half the arc A C which corresponds to it. To prove this, draw the radius O C by which the isosceles triangle O B C is obtained. The angle A O C=O B C+O C B; but O B C=O C B, therefore A O C=2 O B C. Now the angle at the centre A O C is measured by the arc A C, which corresponds to it: consequently the angle at the circumference C B O, which is the half of A O C, is measured by half the arc A C.

Fig. 6.

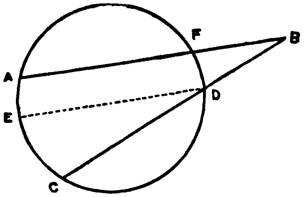


[If, from the extremity C of the chord B C, a perpendicular C D be drawn to A B, B D *is the projection* of the chord B C on A B (79)].

The case of two chords meeting on the circumference, neither of which passes through the centre, is left as an exercise.

137. Let the two secants A B, C B, meet in a point B without the circle.

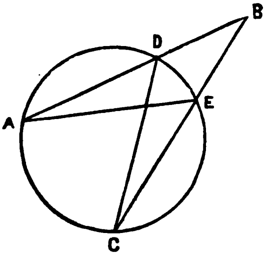
FIG. 157.



Draw E D parallel to A B, the angle E D C=A B C (47). We know (136) that the angle E D C corresponds to half the arc E C which equals $\frac{1}{2}$(arc A C – arc A E)=$\frac{1}{2}$(arc A C – arc D F). Hence the following statement: *The angle formed by two secants which meet without a circle is measured by half the difference of the arcs they intercept.*

138. In fig. 158, if the chords A E, C D be drawn, the triangles A B E, C B D are similar. For the angles A and C are equal, as corresponding to half the common arc D E, and the angle B is common to both triangles.

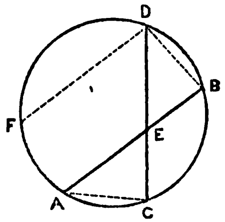
FIG. 158.



139. We shall now consider the case when two chords intersect within the circle.

Let two chords A B, C D (fig. 159), cut one another in the point E. Draw D F parallel to A B. The angle C E A=C D F

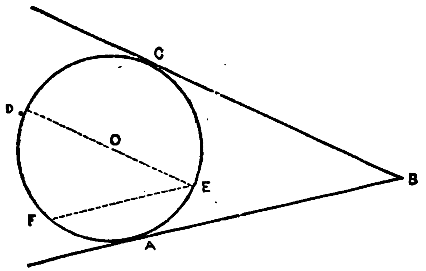
FIG. 159.



(47). But the angle C D F is measured by $\frac{1}{2}$ arc C F, therefore C E A is also measured by $\frac{1}{2}$ arc C F=$\frac{1}{2}$ arc A C+$\frac{1}{2}$ arc A F, but as arc A F=arc B D, consequently the angle C E A is measured by $\frac{1}{2}$ arc A C+$\frac{1}{2}$ arc B D=$\frac{1}{2}$ (arc A C+arc B D). Hence the angle formed by two chords which intersect within a circle is measured by half the sum of the arcs they intercept.

In order to show that the triangles B E D and A E C are similar, it is only necessary to remark that the angles B D E and C A E are equal, each being measured by $\frac{1}{2}$ arc B C, and the angles at the vertex E are equal. The foregoing proof also applies to the case when the two chords intersect at right angles.

FIG. 160.



Let two tangents to the circle O intersect in the point B.

Draw a diameter D E parallel to C B, and from E draw E F parallel to A B.

The arc C D=arc C E, and arc A F=arc A E; therefore arc D F=arc C D F A—arc C E A. But the angle A B C is equal to the angle D E F; consequently the angle formed by the tangents A B, C B is measured by half the difference of the arcs which they intercept.

Two Circles.

140. Two circles may have *internal contact* (fig. 161), or *external contact* (fig. 162). Two circles can only cut one another in two points.

FIG. 161.

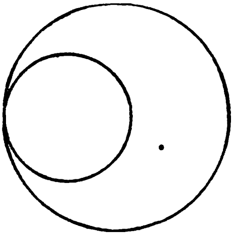
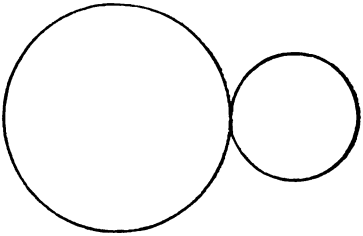
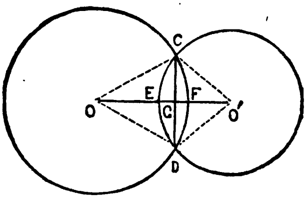


FIG. 162.



Let the circles O and O′, fig. 163, cut one another in the points C and D. Join the centres O and O′ by the straight line O O′, and draw the common chord C D. The *line of centres* O O′ and the chord C D cut one another at right angles. For it is easy to prove that the adjacent angles O G D,

FIG. 163.



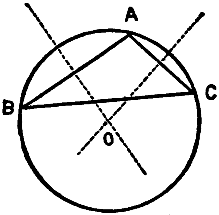
O G C are equal. Further, O C F D is a sector of the larger circle O, and O′C E D is a sector of the smaller circle O′. The *circular segment* C E D F is common to the two sectors, and consists of the sum of the two segments, of which the arc D F C belongs to the larger, and the arc D E C to the smaller circle.

When the two circles are equal, O E=O′F, E C=F C. Arc D E=arc D F=arc F C=arc C E. The triangle O C D =O′C D. Hence O C O′D is a rhombus: the sector O C F D =O′D E C. Segment D F C=segment D E C.

Inscribed and Circumscribed Polygons.

141. Through any three points not in the same straight line one, and only one, circumference can always be drawn.

FIG. 164.



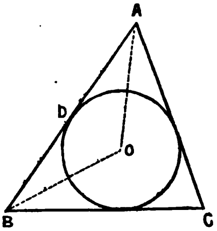
Let A, B, C, fig. 164, be the three points. Draw the straight lines A B, B C, C A, which will be chords of the circle.

The *perpendicular bisectors* of these chords will intersect in a point which is the centre of the circle passing through the three points A, B, C. For the point O is on the perpendicular bisector of A C; it is therefore equally distant from A and C. It is also on the perpendicular bisector of the line A B, and is therefore equally distant from the points A and B. Thus O A=O C=O B=radii of the circle.

The triangle A B C is said to be *inscribed* in the circle. From this it is evident that a circle can always be circumscribed about a triangle.

142. In fig. 165 the circle is said to be *inscribed* in the triangle.

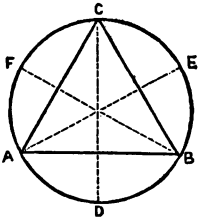
FIG. 165.



Let A B C be a triangle. The three sides must be tangents to the circle. The point of intersection O of the bisectors of the angles A and C will be the centre of the inscribed circle. If from O a perpendicular O D be drawn to A B or B C, it will be a radius of the circle. Thus a circle described with O as centre and radius equal to the perpendicular O D will fulfil the required conditions.

143. We shall now take an equilateral triangle about which a circumference has been circumscribed (fig. 166).

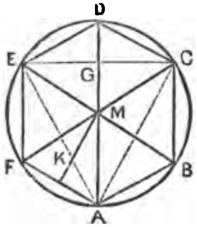
FIG. 166.



The arcs A B, B C, C A are equal, and the diameters drawn from the vertices of the triangle intersect the chords at right angles respectively; consequently arc A D=arc D B=arc B E =arc E C=arc C F=arc F A. If we draw the chords of these arcs, namely, A D, D B, B E, E C, C F, F A, they will be all equal and a regular hexagon will be inscribed in the circle. The angle A D B is measured by $\frac{1}{2}$ arc A C B (136)—that is, $\frac{1}{3} \times 360° = 120°$. D B E $= \frac{1}{2}$ arc D F E $= \frac{360}{3} = 120$, &c.

144. By drawing the three diameters A D, B E, C F (fig. 167) the polygon is divided into six equal and similar isosceles triangles. They are isosceles, for two sides of each are radii of the same circle. And their vertical angles, that is, the angles at the centre M, are measured by equal arcs. Each angle at the point M is $\frac{1}{6} \times 360° = 60°$: this is sufficient to show that each triangle is equilateral. For the two remaining angles are together equal to $180° - 60° = 120°$, and as they are equal angles each is $\frac{1}{2} \times 120° = 60°$. Thus we have shown that the six triangles into which the polygon is divided are equal and equilateral, therefore $AM = AB = BC$, &c., *i.e.* the radius of the circumscribed circle is equal to a side of the regular polygon.

FIG. 167.



If the chords A C, C E, E A (fig. 167) be drawn, the triangle A E C thus formed is equilateral. Each of these chords is bisected at right angles by the diameters E B, A D, C F respectively. We are thus enabled to express a side E C of the inscribed equilateral triangle in terms of the radius of the circumscribing circle.

Let the diameter A D cut the chord E C in G.

$$EG^2 = DG \times AG = DG\,(AD - DG) = DG\,(2EM - DG).$$

But $$EG^2 = ED^2 - DG^2 = EM^2 - DG^2$$

$$\therefore EM^2 - DG^2 = 2DG\,.\,EM - DG^2.$$

or $$EM^2=2DG.EM$$

$$EM=2DG.$$

Thus $$EG^2=DG(2EM-DG)$$
$$=\tfrac{1}{2}EM(2EM-\tfrac{1}{2}EM)$$
$$=EM^2-\frac{EM^2}{4}=\tfrac{3}{4}EM^2$$

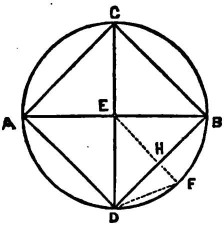
$\therefore EG^2=\frac{3}{4}EM^2$, or $4EG^2=3EM^2$, and $4EG^2=EC^2$, therefore

$$EC=\sqrt{3EM^2}$$
$$=EM\sqrt{3}.$$

Thus if the radius of the circle be known, the side of the inscribed equilateral triangle can be found.

145. If two diameters intersect at right angles, and their extremities be joined by chords, a regular quadrilateral will be inscribed in the circle.

FIG. 168.



The side of the quadrilateral is easily found when the radius of the circle is known. For in the right-angled triangle AEC, $AC^2=AE^2+EC^2=2AE^2 \therefore AC=AE\sqrt{2}$.

By drawing a perpendicular from the centre E on any side BD of the quadrilateral and producing it to meet the circumference in F, the arc BD is divided into two equal parts, and the chord DF is a side of a regular octagon inscribed in the circle. A side of this figure can also be readily found when the radius of the circle is known. For

$$EH=\sqrt{(ED^2-DH^2)}=\sqrt{(ED^2-\tfrac{1}{4}BD^2)}=\sqrt{\left(EC^2-\frac{EC^2}{2}\right)}$$
$$=\sqrt{\tfrac{1}{2}CE^2}$$

thus $FH = CE - \sqrt{\tfrac{1}{2}CE^2} = CE - \tfrac{1}{2}CE\sqrt{2} = \tfrac{1}{2}CE(2-\sqrt{2})$; likewise

$$DF = \sqrt{(DH^2+HF^2)} = \sqrt{\{\tfrac{1}{2}CE^2 + \tfrac{1}{4}CE^2(6-4\sqrt{2})\}}$$
$$= CE\sqrt{(2-\sqrt{2})}$$

The perpendicular EH is termed the *apotome* of the inscribed quadrilateral. In fig. 167 $DG = \tfrac{1}{2}$ radius; thus the apotome of an equilateral triangle inscribed in a circle $=$ radius $- DG = \tfrac{1}{2}$ radius.

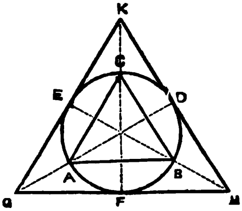
In fig. 168 we found that $HF = \tfrac{1}{2}CE\,(2-\sqrt{2})$; hence the apotome $EH = CE - \tfrac{1}{2}CE\,(2-\sqrt{2}) = \tfrac{1}{2}CE\sqrt{2}$.

In fig. 167 the apotome of the inscribed hexagon $= MK = \sqrt{(FM^2 - FK^2)} = \sqrt{(FM^2 - \tfrac{1}{4}AF^2)} = \sqrt{(FM^2 - \tfrac{1}{4}FM^2)} = \sqrt{\tfrac{3}{4}FM^2} = \tfrac{1}{2}FM\sqrt{3}$.

By producing the apotome of any inscribed polygon until it meets the circle, we are enabled to form an inscribed polygon of twice the number of sides. The point in which the apotome meets the circle must be joined by straight lines to the extremities of the side on which it is drawn.

146. When an inscribed figure is given, a similar figure can always be described about a circle.

Fig. 169.

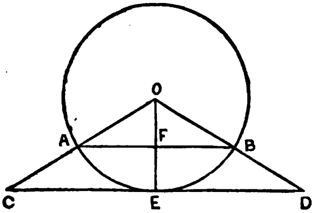


Let ABC (fig. 169) be an equilateral triangle inscribed in a circle. From each vertex draw the perpendicular bisector of the opposite side and produce it to meet the circle in the points D, E, F respectively. The intersections of the tangents at these points are the vertices of an equilateral triangle G, H, K described about the circle.

In a similar manner a polygon of four, five, six, eight, &c., sides can be described about a circle when the corresponding inscribed polygon is given.

147. We can now express a side of the equilateral triangle described about a circle in terms of the radius of the inscribed circle.

FIG. 170.



Let A B (fig. 170) be a side of the inscribed triangle, and C D a side of the triangle described about the circle. The two triangles A O B, C O D are similar; hence

$$CD : AB = OE : OF$$

or

$$OF : OE = AB : CD$$

It follows from (144) that $AB = OA\sqrt{3}$, and $OF = \frac{1}{2} OE$; hence the second proportion may be written

$$\tfrac{1}{2} OF ; OE = AO\sqrt{3} : CD$$
$$\therefore \quad CD = 2\,OA\sqrt{3} = 2\,AB.$$

Thus we see that the side of the equilateral triangle described about the circle is equal to twice the side of the inscribed equilateral triangle.

148. If A B be a side of an inscribed regular quadrilateral (square), then $AB = AE\sqrt{2}$, and the apotome $EF = \frac{1}{2} AE\sqrt{2}$; hence the proportion

$$\tfrac{1}{2} AE\sqrt{2} : AE = AE\sqrt{2} : CD$$
$$\therefore \quad CD = 2\,AE.$$

From this we learn that the side of a quadrilateral described about a circle is equal to the diameter of the latter.

149. If A B is the side of a regular inscribed hexagon,

then $\mathrm{AB}=\mathrm{AE}=$radius, and $\mathrm{EF}=\frac{1}{2}\mathrm{AE}\sqrt{3}$, and the proportion becomes

$$\tfrac{1}{2}\mathrm{AE}\sqrt{3} : \mathrm{AE}=\mathrm{AE} : \mathrm{CD}$$

$$\therefore\quad \mathrm{CD}=\frac{\mathrm{AE}}{\frac{1}{2}\sqrt{3}}=\tfrac{2}{3}\mathrm{AE}\sqrt{3}.$$

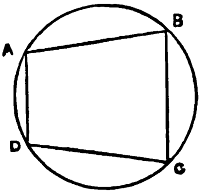
150. If AB is the side of a regular inscribed octagon, then $\mathrm{AB}=\mathrm{AE}\sqrt{(2-\sqrt{2})}$, and $\mathrm{EF}=$apotome$=\frac{1}{2}\ \mathrm{AE}\sqrt{(2+\sqrt{2})}$. The proportion is

$$\tfrac{1}{2}\mathrm{AE}\sqrt{\ 2+\sqrt{2})} : \mathrm{AE}=\mathrm{AE}\sqrt{(2-\sqrt{2})} : \mathrm{CD}$$

$$\therefore\qquad \mathrm{CD}=2\,\mathrm{AE}\,(3-2\sqrt{2}).$$

151. If any four points be taken on the circumference of a circle and joined by straight lines so as to form a quadrilateral ABCD (fig. 171), then the angle BAD is measured by $\frac{1}{2}$ arc BCD, and the angle BCD is measured by $\frac{1}{2}$ arc BAD; thus the sum of the opposite angles BAD, $\mathrm{BCD}=\frac{1}{2}$ (arc BCD +arc $\mathrm{BAD})=\frac{1}{2}\times 360^\circ=180^\circ=2$ right angles. And likewise the angle ABC is measured by $\frac{1}{2}$ arc ADC, and the angle ADC is measured by $\frac{1}{2}$ arc ABC; thus,

Fig. 171.



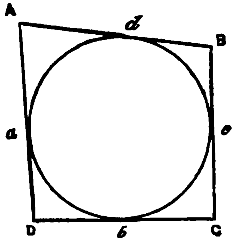
Angle ABC+angle $\mathrm{ADC}=\frac{1}{2}$ (arc ADC+arc $\mathrm{ABC})=\frac{1}{2}\times 360^\circ$ $=180^\circ=2$ right angles.

Hence we may conclude that when the sum of two opposite angles of any quadrilateral is equal to two right angles, a circle can be circumscribed about it.

N.B. A circle can be circumscribed about every square, rectangle, and isosceles trapezoid, but not about every parallelogram, rhombus, and trapezium.

152. Let four tangents be drawn to a circle which intersect so as to form a quadrilateral (fig. 172).

FIG. 172.



Let a, b, c, d be the points of contact of the tangents and circle respectively. Then we know (Art. 142) that

$$\begin{array}{c} a\mathrm{D}=\mathrm{D}b \\ \mathrm{C}c=\mathrm{C}b \\ \mathrm{B}c=\mathrm{B}d \\ \underline{\mathrm{A}a=\mathrm{A}d} \end{array}$$

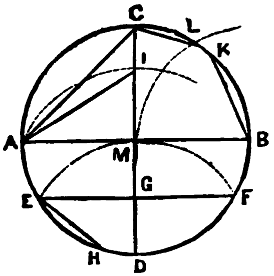
thus $a\mathrm{D}+\mathrm{C}c+\mathrm{B}c+\mathrm{A}a=\mathrm{D}b+\mathrm{C}b+\mathrm{B}d+\mathrm{A}d$, that is, $\mathrm{AD}+\mathrm{CB}=\mathrm{DC}+\mathrm{AB}$.

The sides of every quadrilateral described about a circle are evidently tangents to that circle; and we have just seen that the sums of the opposite sides of such a quadrilateral are equal. Thus a circle can be inscribed in every square and rhombus, but not in every rectangle, parallelogram, and isosceles trapezoid, or, in fact, in any quadrilateral, the sums of whose opposite sides are not equal.

153. We shall conclude this chapter by showing how regular polygons of from three to twelve sides may be inscribed in a circle.

Draw two diameters A B, D C (fig. 173) at right angles to one another. A C is a side of an inscribed square. With D as centre and radius equal to D M describe an arc E M F, and draw the chord E F. Then E F is a side of the inscribed equilateral triangle; D F is a side of the inscribed hexagon. Half the chord E F, that is, E G is a side of the inscribed heptagon. The chord E H which subtends a third of the arc E D F is a side of the inscribed nonagon.

FIG. 173.

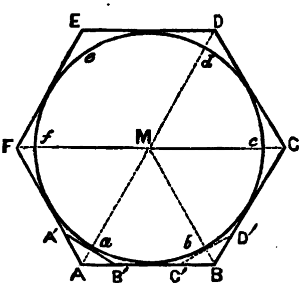


With G as centre and G A as radius describe the arc A I cutting the diameter C D in I; then M I is a side of the inscribed regular decagon. A I is a side of the inscribed pentagon. If B C be bisected in K, the chord B K is a side of the inscribed octagon. Lastly, with B as centre and B M as radius describe an arc M L; then L C is the side of the inscribed dodecagon, and I L is the side of the inscribed undecagon or eleven-sided polygon.

154. We have shown in the last chapter how a polygon can be inscribed in a circle. Let us suppose a polygon of n sides to be inscribed. It is evident that the sum of these sides is the perimeter of the polygon, and that this perimeter must be less than the circumference of the circumscribing circle. If we now double the number of sides, that is, inscribe a polygon of $2n$ sides, the perimeter of this second polygon, though greater than the perimeter of the first, is still less than the circumference of the circumscribing circle. Thus we see that as the number of sides of each succeeding polygon is increased, its perimeter approaches closer and closer to the circumference of the circumscribing circle.

Let ABCDEF be a regular polygon described about a circle. Draw AM, BM, CM, &c., and through the points *a*, *b*, *c*, *d*, &c., draw tangents C′D′, A′B′, E′F′, &c. D′C′B′A′E′, &c., will be a new polygon described about the circle of twice the number of sides of the first.

FIG. 174.



If this process were repeated a few times, the result would be a described polygon of almost an infinite number of sides.

155. We see from fig. 167 that the perimeter of the inscribed polygon consists of a certain number of equal chords, and that the circumference of the circle consists of the same number of equal arcs; and as the length of each arc is greater than its chord, it follows that the perimeter of the inscribed polygon must be less than the circumference of the circumscribing circle.

As we increase the number of sides of each succeeding regular polygon described about the same circle, we know that its perimeter becomes less and less, and that it eventually approaches very closely indeed to the circle itself. Yet, however great the number of sides may be, the perimeter of the polygon described about the circle must, mathematically considered, be always greater than the circumference of the circle.

If we continue to double the number of sides of the inscribed and described polygons simultaneously, the perimeters of each succeeding pair of polygons will approach each other, and consequently also the circumference of the circle closer and closer. And we may take the number of sides of the two polygons so great that the difference between their perimeters will be as small as any quantity we please; but smaller still will be the difference between the circumference of the circle and the perimeter of either of the two polygons.

156. In order to find the ratio of the diameter of a circle to its circumference, we first find the length of a side of the n-sided polygon inscribed in, and also the length of a side of the n-sided polygon described about the same circle. Then we proceed to find a side of the $2n$-sided polygon inscribed in, and also a side of the $2n$-sided polygon described about the same circle. These calculations are to be continued until the difference between the perimeters of the two corresponding polygons is less than the degree of exactitude to which the ratio of the circumference of the circle to its diameter is required. If these perimeters differ only in the fourth figure after the decimal point, then we know that each of them differs from the circumference by less than one-thousandth part of the radius. This will be better understood by considering the following example. Take a regular hexagon inscribed in a circle whose diameter$=2$.

Referring to fig. 170

$$\mathrm{OF} : \mathrm{OE} = \mathrm{AB} : \mathrm{CD}$$

or

$$\sqrt{(\mathrm{AO}^2 - \tfrac{1}{4}\mathrm{AB}^2)} : \mathrm{OE} = \mathrm{AB} : \mathrm{CD}$$

or

$$\mathrm{CD} = \frac{\mathrm{OE} \times \mathrm{AB}}{\sqrt{(\mathrm{AO}^2 - \tfrac{1}{4}\mathrm{AB}^2)}} = \frac{1}{\sqrt{(1 - \tfrac{1}{4})}} = \frac{1}{\tfrac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

Now we know that the perimeter of the inscribed hexagon is $6 \times 1 = 6$, and we learn from the above calculation that the perimeter of a hexagon described about the same circle $= 6 \times \frac{2}{3}\sqrt{3} = 4\sqrt{3} = 6{\cdot}928203203755$. Between these two perimeters the circumference of the circle will be found; but we must have a closer relation than this—that is, greater accuracy.

Proceeding in this way we shall obtain the following results :—

No. of sides of inscribed polygon.	Perimeter.
12.	=6·2116570
24.	=6·2652572
48.	=6·2787004
96.	=6·2820639
192.	=6·2829049
384.	=6·2831152

No. of sides of polygon described about the same circle.	Perimeter.
6.	=6·9282032
12.	=6·4307806
24.	=6·3193199
48.	=6·2921724
96.	=6·2854292
192.	=6·2837461
384.	=6·2833260

These results clearly show that as the number of sides of a corresponding pair of polygons is increased, their perimeters approach each other closer and closer. The perimeters of the 48-sided polygons differ only in the tenths, those of the 96-sided in the hundredths, and those of the 384-sided in the thousandths.

157. Archimedes, who stopped with the 96-sided polygons, found that the circumference of the circle was larger than $6\frac{20}{71}$ and smaller than $6\frac{20}{70}$; from this he derived the well-known ratio 7 : 22—that is, if the diameter of a circle be divided into 7 equal parts, its circumference will contain 22 of those parts, nearly. This is the ratio of the diameter of a circle to its circumference. It can never be expressed in finite numbers. Other ratios have been calculated by mathematicians. A. Metius, a Dutch mathematician, found

a nearer ratio, namely 113 : 355. This is easily remembered by writing down the first three odd numbers in pairs, and then halving them; thus, 113|355. Ludolf van Keule, another Dutch mathematician, calculated the ratio to 36 figures. He found that if the diameter be equal to unity, then the circumference is

3·14159,26535,89793,23846,26433,83279,50288.

In practice, however, we only use the first six figures of this expression. Thus, assuming the diameter of a circle$=1$, its circumference$=3·14159$. This latter number is commonly denoted by the Greek letter π.

158. If the number of sides of a polygon be very great, it will be readily conceded that the perimeter of the polygon and the circumference of the circle in which it is inscribed are practically the same; in fact, the one merges into the other, and the apotome of the polygon becomes equal to the radius of the circle. Thus we obtain the following expression for finding the area of a circle—

Area of polygon=perimeter × $\frac{1}{2}$ apotome.
Area of circle=circumference × $\frac{1}{2}$ radius.

If we denote the radius of a circle by r, then the diameter$=2\,r$, and the circumference$=2\,\pi\,r$. Thus the area of the circle$=\pi\,r^2$=half the circumference multiplied by the radius.

If d denote the diameter, then the radius$=\frac{1}{2}\,d$, the circumference$=\pi\,d$, and area$=\frac{1}{4}\,\pi\,d^2$.

If the area of a given circle$=1$, then

$$\text{the radius}=\sqrt{\frac{1}{\pi}}$$

$$\text{diameter}=2\sqrt{\frac{1}{\pi}}$$

$$\text{circumference}=2\,\pi\sqrt{\frac{1}{\pi}}=2\sqrt{\pi}$$

159. Lastly, let g denote the number of degrees measured

by an arc of a circle, l the length of that arc, and c the circumference of the circle, then we have the following proportion:

$$g : 360^\circ :: l : c$$

but we have found that $c = 2\pi r$ (158),

or $$g : 360^\circ = l : 2\pi r$$

$$\therefore \quad g : \frac{180^\circ}{\pi} = l : r$$

Hence we can find the length of an arc of a circle, if the radius and number of degrees in that arc are known.

160. We can also find the area of the sector of a circle. In fig. 153 the sector AOB is contained as many times in the whole circle as the arc AB is contained in the whole circumference. Thus we have

$$\frac{1}{\text{sector AOB}} = \frac{360^\circ}{g} \text{ or sector AOB} : \pi r^2 = g : 360^\circ$$

or $$\text{sector AOB} : \pi r^2 = \frac{180^\circ \times l}{\pi r} : 360^\circ$$

$$\text{sector AOB} = \frac{lr}{2} = l \times \tfrac{1}{2} r$$

that is, the area of the sector AOB is equal to the length of the arc multiplied by half the radius. From this we find that

$$\text{area of segment AB} = \text{area segment AOB} - \triangle \text{AOB}.$$

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